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Abstract

Full Text

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ASYMPTOTIC EXPRESSION FOR THE CONFLUENT HYPERGEOMETRIC FUNCTION

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The asymptotic behavior of solutions of a differential equation of the form

$$u'' + f(x)u = 0, \tag{1}$$

where $f(x)$ is a function of one or several parameters, can be studied by constructing a so-called "nearby equation," on the basis that differential equations which are approximately the same for all x must have approximately the same solutions. We shall give a method for constructing such an equation; in particular, we shall apply the results obtained to finding asymptotic formulas for Whittaker' s confluent hypergeometric function $W_{\lambda,\mu}(x)$ [(1)]. As is known, the Hermite and Laguerre polynomials, as well as the Bessel functions, are special cases of this function.

We shall seek the solution of equation (1) in the form:

$$u = \psi(x)F[z(x)], \tag{2}$$

where φ, F, and z are arbitrary functions. Substituting (2) into (1), we obtain:

$$\left\{F'' + \frac{1}{z}F' + F\left(1 - \frac{s^2}{z^2}\right)\right\} + F'\left\{\frac{2\psi'}{\psi z'} + \frac{z''}{z'^2} - \frac{1}{z}\right\} +$$

$$+F\left\{\frac{\psi''}{\psi z'^2} + \frac{f}{z'^2} - \left(1 - \frac{s^2}{z^2}\right)\right\} = 0, \tag{3}$$

where $'$ denotes differentiation with respect to the argument, and the parameter s will be determined later. It should be noted that in (3) we have proceeded along the path of constructing Bessel' s equation for the function $F(z)$. Equating to zero each of the expressions in braces, we obtain:

$$u = \left(\frac{z}{z'}\right)^{1/2}\left\{AZ_s^{(1)}(z) + BZ_s^{(2)}(z)\right\}. \tag{4}$$

Here $Z_s^{(1)}$ and $Z_s^{(2)}$ are two linearly independent solutions of Bessel' s equation, A and B are constants, and z is determined by the equation

$$z'^2(1+\varepsilon) = f^*, \tag{5}$$

where

$$\varepsilon = \frac{z'''}{2z'^3} - \frac{3z''^2}{4z'^4} + \frac{1}{z^2}\left(\frac{1}{4} - s^2\right) - \frac{f - f^*}{z'^2}. \tag{6}$$

The parameter s and the arbitrary function f^* are chosen in such a way that over the entire interval of variation of x the quantity ε remains much less than unity. It can be shown that when $f = f^*$ the asymptotic formulas are expressed through Bessel functions of order $s = \pm\frac{1}{m+2}$, where m is the multiplicity of the zero of the equation $f(x_0) = 0$. Near the root $x = x_0(1+\xi)$, where $\xi \to 0$,

the quantity

$$\varepsilon \sim \frac{1}{\xi^{m+2}}\left[\frac{1}{m+2} - s^2\right]^{(2-4)},$$

whence we find the condition under which ε remains much smaller than unity.

In particular, if x_0 is a simple root of the equation $f(x_0) = 0$, the asymptotic solutions are expressed in terms of Bessel functions of order $\pm 1/3$. Another choice of f^* ensures the condition $\varepsilon \ll 1$ for a corresponding choice of the parameter s, which makes it possible to obtain asymptotic formulas of Hilb type ([5],[6]). Neglecting ε in equation (5), we obtain for z:

$$z = \int (f^*)^{1/2}\, dx. \tag{7}$$

As is known, Whittaker' s equation has the form:

$$\frac{d^2W}{dx^2} + \left\{-\frac{1}{4} + \frac{\lambda}{x} + \frac{1/4 - \mu^2}{x^2}\right\} W = 0. \tag{8}$$

Our task is to obtain asymptotic solutions of equation (8) for large λ and fixed μ, valid uniformly for all values of x from 0 to ∞. Choosing $f^* = -1/4 + 1/4$, it is necessary to consider three ranges of variation of x:

$$\text{I.}\quad 0 \le x \le x^* < 4\lambda, \quad z_1 = \int_0^x \left(\frac{\lambda}{x} - \frac{1}{4}\right)^{1/2} dx, \quad s = 2\mu;$$

$$\text{II.}\quad x^* \le x \le 4\lambda, \quad z_2 = \int_x^{4\lambda} \left(\frac{\lambda}{x} - \frac{1}{4}\right)^{1/2} dx, \quad s = 1/3;$$

$$\text{III.}\quad 4\lambda \le x < \infty, \quad z_3 = \int_{4\lambda}^x \left(\frac{1}{4} - \frac{\lambda}{x}\right)^{1/2} dx, \quad s = 1/3.$$

Here x^* is some interior point of the interval $0, 4\lambda$. According to (4), the asymptotic solutions for each of the three ranges take the form:

$$W_{\lambda,\mu}(x) = \left(\frac{z_1}{z_1'}\right)^{1/2} \{A_1 J_{2\mu}(z_1) + B_1 N_{2\mu}(z_1)\}, \qquad 0 \le x \le x^* < 4\lambda, \tag{9}$$

$$W_{\lambda,\mu}(x) = \left(\frac{z_2}{-z_2'}\right)^{1/2} \{A_2 J_{1/3}(z_2) + B_2 J_{-1/3}(z_2)\}, \qquad x^* \le x \le 4\lambda, \tag{10}$$

$$W_{\lambda,\mu}(x) = \left(\frac{z_3}{z_3'}\right)^{1/2} \{A_3 I_{1/3}(z_3) + B_3 K_{1/3}(z_3)\}, \qquad 4\lambda \le x < \infty. \tag{11}$$

The constants A_3 and B_3 are easily determined by using the behavior of $W_{\lambda,\mu}(x)$, $I_{1/3}(z_3)$, and $K_{1/3}(z_3)$ at infinity, which gives:

$$A_3 = 0, \qquad B_3 = \frac{1}{\sqrt{\pi}} e^{-\lambda + \lambda \ln \lambda}. \tag{12}$$

Comparing (10) and (11) as $x \to 4\lambda$, we obtain:

$$A_2 = B_2 = \sqrt{\frac{\pi}{3}}\, e^{-\lambda + \lambda \ln \lambda}. \tag{13}$$

To determine the coefficients A_1 and B_1, we require equality of the asymptotic solutions (9) and (10) and their derivatives at some interior point of the interval $0, 4\lambda$, whence, taking into account that $z_1 + z_2 = \pi\lambda$, we have

$$A_1 = \sqrt{\pi}\, e^{-\lambda + \lambda \ln \lambda} \sin(\lambda - \mu)\pi; \qquad B_1 = -\sqrt{\pi}\, e^{-\lambda + \lambda \ln \lambda} \cos(\lambda - \mu)\pi. \tag{14}$$

Finally, the required asymptotic formulas take the form:

$$W_{\lambda,\mu}(x) = \sqrt{\pi}\, e^{-\lambda+\lambda\ln\lambda} \left(\frac{z_1}{z_1'}\right)^{1/2} \{\sin(\lambda-\mu)\pi J_{2\mu}(z_1) - \cos(\lambda-\mu)\pi N_{2\mu}(z_1)\}, \tag{15}$$

$$z_1 = 2\lambda \arcsin\left(\frac{x}{4\lambda}\right)^{1/2} + 2\lambda\left(\frac{x}{4\lambda}\right)^{1/2}\left(1-\frac{x}{4\lambda}\right)^{1/2}, \qquad 0 \le x \le x^* < 4\lambda;$$

$$W_{\lambda,\mu}(x) = \sqrt{\frac{\pi}{3}}\, e^{-\lambda+\lambda\ln\lambda} \left(\frac{z_2}{-z_2'}\right)^{1/2} \{J_{1/3}(z_2) + J_{-1/3}(z_2)\},$$

$$z_2 = 2\lambda \arccos\left(\frac{x}{4\lambda}\right)^{1/2} - 2\lambda\left(\frac{x}{4\lambda}\right)^{1/2}\left(1-\frac{x}{4\lambda}\right)^{1/2}, \qquad x^* \le x \le 4\lambda; \tag{16}$$

$$W_{\lambda,\mu}(x) = \frac{1}{\sqrt{\pi}}\, e^{-\lambda+\lambda\ln\lambda} \left(\frac{z_3}{z_3'}\right)^{1/2} K_{1/3}(z_3),$$

$$z_3 = 2\lambda\left(\frac{x}{4\lambda}\right)^{1/2}\left(\frac{x}{4\lambda}-1\right)^{1/2} - 2\lambda\ln\left\{\left(\frac{x}{4\lambda}\right)^{1/2} - \left(\frac{x}{4\lambda}-1\right)^{1/2}\right\}, \qquad 4\lambda \le x < \infty. \tag{17}$$

The arguments in formulas (15)–(17) can be simplified by introducing a new variable $x = 4\lambda \sin^2\frac{\varphi}{2}$ in (15), $x = 4\lambda\cos^2\frac{\varphi}{2}$ in (16), and $x = 4\lambda\,\mathrm{ch}^2\,\frac{\Phi}{2}$ in (17). Then (15)–(17) pass respectively into (18)–(20):

$$W_{\lambda,\mu}\left(4\lambda\sin^2\frac{\varphi}{2}\right) = \sqrt{2\pi\lambda}\, e^{-\lambda+\lambda\ln\lambda}\left(\frac{\varphi+\sin\varphi}{\mathrm{ctg}(\varphi/2)}\right)^{1/2} \times$$

$$\times\{\sin(\lambda-\mu)\pi J_{2\mu}(\lambda\varphi+\lambda\sin\varphi) - \cos(\lambda-\mu)\pi N_{2\mu}(\lambda\varphi+\lambda\sin\varphi)\}, \tag{18}$$

$$0 \le \varphi \le \varphi^* < \pi;$$

$$W_{\lambda,\mu}\left(4\lambda\cos^2\frac{\varphi}{2}\right) = \sqrt{\frac{2\pi\lambda}{3}}\, e^{-\lambda+\lambda\ln\lambda}\left(\frac{\varphi-\sin\varphi}{\mathrm{tg}(\varphi/2)}\right)^{1/2} \times$$

$$\times\{J_{1/3}(\lambda\varphi-\lambda\sin\varphi) + J_{-1/3}(\lambda\varphi-\lambda\sin\varphi)\}, \qquad 0 \le \varphi \le \varphi^* < \pi; \tag{19}$$

$$W_{\lambda,\mu}\left(4\lambda\operatorname{ch}^2\frac{\Phi}{2}\right)=\sqrt{\frac{2\lambda}{\pi}}\,e^{-\lambda+\lambda\ln\lambda}\left(\frac{\operatorname{sh}\Phi-\Phi}{\operatorname{th}(\Phi/2)}\right)^{1/2}K_{1/3}(\lambda\operatorname{sh}\Phi-\lambda\Phi),\quad(20)$$

$$0\leq\Phi<\infty.$$

If we assume that $4\lambda\gg x^*$, (15) is simplified and takes the form

$$W_{\lambda,\mu}(x)=\sqrt{2\pi x}\,e^{-\lambda+\lambda\ln\lambda}\{\sin(\lambda-\mu)\pi J_{2\mu}(2\sqrt{\lambda x})-$$

$$-\cos(\lambda-\mu)\pi N_{2\mu}(2\sqrt{\lambda x})\}.\quad(21)$$

This is an asymptotic formula of Hilb type.

From formulas (15), as well as (18) and (21), it is seen that Whittaker's equation will have a solution bounded at zero if the coefficient of $N_{2\mu}(z)$ vanishes, i.e. $\lambda=l+\mu+\frac{1}{2}$, where $l+1$ is a natural number. This makes it possible to determine exactly the eigenvalues of the energy operator of the hydrogen atom $E_n=-\frac{mZ^2e^4}{2\hbar^2n^2}$, $n=1,2,...$ (see (22)) and of the harmonic oscillator $E_n=(n+\frac{1}{2})\hbar\omega$, $n=0,1,2,...$ (see (26)).

Taking into account the relation of Laguerre polynomials with the Whittaker function

$$L_l^\alpha(x)=(-1)^l x^{-\frac{\alpha+1}{2}}e^{x/2}W_{l+\frac{\alpha+1}{2},\,\frac{\alpha}{2}}(x),\quad(22)$$

we obtain the following formulas, valid for large l:

$$L_l^\alpha\left(4\lambda\sin^2\frac{\varphi}{2}\right)=\sqrt{\pi}\,\frac{\exp[-\lambda\cos\varphi+(\lambda-\alpha/2)\ln\lambda]}{(2\sin(\varphi/2))^\alpha}\times$$

$$\times(\varphi\csc\varphi+1)^{1/2}J_\alpha(\lambda\varphi+\lambda\sin\varphi),\qquad 0\leq\varphi\leq\varphi^*<\pi;\quad(23)$$

$$L_l^\alpha\left(4\lambda\cos^2\frac{\varphi}{2}\right)=(-1)^l\sqrt{\frac{\pi}{3}}\,\frac{\exp[\lambda\cos\varphi+(\lambda-\alpha/2)\ln\lambda]}{(2\cos(\varphi/2))^\alpha}\times$$

$$\times(\varphi\csc\varphi-1)^{1/2}\left\{J_{1/3}(\lambda\varphi-\lambda\sin\varphi)+J_{-1/3}(\lambda\varphi-\lambda\sin\varphi)\right\},\quad 0\leqslant\varphi\leqslant\varphi^*<\pi;\quad(24)$$

$$L_l^\alpha\left(4\lambda\operatorname{ch}^2\frac{\Phi}{2}\right)=\frac{(-1)^l}{\sqrt{\pi}}\,\frac{\exp[\lambda\operatorname{ch}\Phi+(\lambda-\alpha/2)\ln\lambda]}{(2\operatorname{ch}(\Phi/2))^\alpha}\times$$

$$\times (1 - \Phi \operatorname{csch} \Phi)^{1/2} K_{1/3}(\lambda \operatorname{sh} \Phi - \lambda \Phi), \quad 0 \leqslant \Phi < \infty. \tag{25}$$

In the last three formulas $\lambda = l + \dfrac{\alpha + 1}{2}$. The case of large α and fixed l was considered in ([7]). For $\alpha = \pm 1/2$, the Laguerre polynomials are simply related to the Hermite polynomials

$$\mathcal{H}_{2l}(x) = (-1)^l 2^{2l} L_l^{-1/2}(x^2), \qquad \mathcal{H}_{2l+1}(x) = (-1)^l 2^{2l+1} x L_l^{1/2}(x^2), \tag{26}$$

and from formulas (23)–(25) we obtain

$$H_n\left(\sqrt{2n+1}\sin\frac{\varphi}{2}\right) = \frac{(2n+1)^{n/2}}{(2\cos(\varphi/2))^{1/2}} \exp\left[-\frac{2n+1}{4}\cos\varphi\right] \times$$
$$\times \cos\left\{\frac{2n+1}{4}(\varphi + \sin\varphi) - \frac{\pi n}{2}\right\}, \quad 0 \leqslant \varphi \leqslant \varphi^* < \pi; \tag{27}$$

$$H_n\left(\sqrt{2n+1}\cos\frac{\varphi}{2}\right) = \frac{1}{2}\sqrt{\frac{\pi}{3}}\,(2n+1)^{\frac{n+1}{2}} \exp\left[\frac{2n+1}{4}\cos\varphi\right]\left(\frac{\varphi - \sin\varphi}{\sin(\varphi/2)}\right)^{1/2} \times$$
$$\times \left\{J_{1/3}\left[\frac{2n+1}{4}(\varphi - \sin\varphi)\right] + J_{-1/3}\left[\frac{2n+1}{4}(\varphi - \sin\varphi)\right]\right\}, \quad 0 \leqslant \varphi \leqslant \varphi^* < \pi; \tag{28}$$

$$H_n\left(\sqrt{2n+1}\operatorname{ch}\frac{\Phi}{2}\right) = \frac{(2n+1)^{\frac{n+1}{2}}}{2\sqrt{\pi}} \exp\left[\frac{2n+1}{4}\cos\Phi\right]\left(\frac{\operatorname{sh}\Phi - \Phi}{\operatorname{sh}(\Phi/2)}\right)^{1/2} \times$$
$$\times K_{1/3}\left[\frac{2n+1}{4}(\operatorname{sh}\Phi - \Phi)\right], \quad 0 \leqslant \Phi < \infty. \tag{29}$$

In the case of the Laguerre polynomials, (21) passes into the well-known Hilb-type formula belonging to Szegő:

$$L_l^{\alpha}(x) = \sqrt{2\pi}\exp\left[-\lambda + \lambda\ln\lambda + \frac{x}{2}\right] x^{-\alpha/2} J_\alpha(2\sqrt{\lambda x}), \tag{30}$$

and in the case of the Hermite polynomials one obtains Adamov' s formula

$$H_n(x) = \frac{(2n+1)^{n/2}}{\sqrt{2}} \exp\left[-\frac{2n+1}{4} + \frac{x^2}{2}\right] \cos\left(\sqrt{2n+1}\,x - \frac{\pi n}{2}\right). \tag{31}$$

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CITED LITERATURE

1. E. T. Whittaker, G. N. Watson, *A Course of Modern Analysis*, 2, Moscow—Leningrad, 1934.

2. A. A. Sokolov, *Vest. MGU*, No. 4, 77 (1947).

3. D. Ivanenko, A. Sokolov, *Classical Field Theory*, Moscow—Leningrad, 1951.

4. R. E. Langer, *Trans. Am. Math. Soc.*, **33**, 23 (1931); **34**, 447 (1932); R. E. Langer, *Phys. Rev.*, **51**, 669 (1937).

5. E. Hilb, *Math. Zs.*, **5** (1919).

6. A. A. Sokolov, B. K. Kerimov, *DAN*, **108**, No. 4, 611 (1956); A. Sokolov, B. Kerimov, *Nuovo Cim.*, No. 5, 921 (1957); R. M. Muradyan, *DAN*, **115**, No. 5, 887 (1957).

7. A. A. Sokolov, N. P. Klepikov, I. M. Ternov, *ZhETF*, **24**, 249 (1953).

Note: Figure translations are in progress. See original paper for figures.