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Abstract

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ON THE CONVERGENCE OF CERTAIN FINITE-DIFFERENCE PROCESSES FOR THE EQUATIONS $y' = f(x,y)$ AND $y'(x) = f(x, y(x), y(x-\tau(x)))$

(Presented by Academician S. L. Sobolev on 22 XI 1957)

An extensive literature is devoted to the description and study of finite-difference methods for solving the Cauchy problem for the equations indicated in the title (see, for example, ($^{1-6}$)). In the present article the convergence of certain finite-difference processes for solving the Cauchy problem for such equations is investigated, and some estimates are obtained.

1°. Suppose the equation

$$y' = f(x,y), \tag{1}$$

is given, whose right-hand side is defined and continuous in a certain bounded closed domain $\overline{G}$ of the (x,y)-plane; it is required to find its solution satisfying the condition

$$y(x_0) = y_0, \qquad (x_0, y_0) \in G. \tag{2}$$

By $y = y(x)$ we shall denote any exact solution of problem (1), (2) (^{7}).

We shall seek approximate values y_i of the ordinates $y(x_i)$ of the exact solution $y = y(x)$ at the points $x_i = x_0 + ih,\ i = 0, \pm 1, \pm 2, \dots;\ h > 0$, by means of the finite-difference equation:

$$\sum_{i=0}^{m} \alpha_i y_{k-i} = h \sum_{i=0}^{n} \beta_i f_{k+l-i}, \qquad f_j = f(x_j, y_j), \tag{3}$$

where l, m, and n are given integers; α_i and β_i are specified real numbers; $m > 0$; $n \geq 0$; $\alpha_0, \alpha_m, \beta_0, \beta_n$ are nonzero (^{6}).

Noting that the order of equation (3) is equal to $p - q$, where $p = \max(k, k+l)$ and $q = \min(k-m, k+l-n)$, we prescribe the initial conditions in the form

$$y_i = g(x_i), \qquad i = 0, -1, \dots, -(p-q-1), \qquad g(x_0) = y_0, \tag{4}$$

where $g(x)$ is a certain continuously differentiable function, given on the interval $x^* \leq x \leq x_0$ and called the initial function; here the interval $[x^*, x_0]$ constitutes part of the projection of the intersection of the straight line $y = y_0$ with the domain G onto the x-axis, and $(x, g(x)) \in G$; the possibility is allowed that the initial function may change when the step h is changed, which must satisfy the condition $h(p-q-1) \leq x^* - x_0$.

Suppose that problem (3), (4) has been solved on some interval of values of x. Join each pair of points (x_j, y_j) and (x_{j+1}, y_{j+1}) by a line segment, and let the equation of the polygonal line thus obtained be $y = \tilde{y}_h(x)$. Suppose, further, that the numbers $\overline{x}$ and h_0 are chosen so that the mentioned polygonal line is defined, in any case, on the interval $x_0 \leq x \leq \overline{x}$ for every h, $0 < h \leq h_0$. If from every sequence of steps converging to zero one can extract a subsequence h_ν, $\nu = 1, 2, \dots$, such that the corresponding sequence 0

of the corresponding broken lines $y = \tilde{y}_{h_\nu}(x)$ converges uniformly on the interval $[x_0, \bar{x}]$ to some solution of problem (1), (2), then we shall say that the finite-difference process defined by equation (3) converges.

Theorem 1. *If the finite-difference process defined by equation (3) converges, then the conditions*

$$\sum_{i=0}^{m} \alpha_i = 0, \qquad \sum_{i=0}^{m-1} \sum_{j=0}^{i} \alpha_j = \sum_{i=0}^{n} \beta_i \neq 0 \tag{5}$$

are satisfied.

2°. Denote by $C_h[x_0, \bar{x}]$ the space of finite functions $\psi(x_k)$, given at the points x_k, $k = 0, 1, \dots, N_h = \left[\dfrac{\bar{x} - x_0}{h}\right]$, with norm

$$\|\psi\| = \max_{0 \leq k \leq N_h} |\psi(x_k)|,$$

and by $C[x_0, \bar{x}]$ the space of bounded functions $\psi(x)$, given on the interval $x_0 \leq x \leq \bar{x}$, with norm

$$\|\psi\| = \sup_{x_0 \leq x \leq \bar{x}} |\psi(x)|.$$

The space $C_h[x_0, \bar{x}]$ can be embedded in $C[x_0, \bar{x}]$, identifying elements of $C_h[x_0, \bar{x}]$ with the corresponding piecewise constant functions from $C[x_0, \bar{x}]$.

Consider the finite-difference equation

$$\sum_{i=0}^{m-1} \sum_{j=0}^{i} \alpha_j \varphi_{k-i} = h\psi_k, \tag{6}$$

where $\varphi_j = \varphi(x_j)$, $\psi_j = \psi(x_j)$. Under zero initial conditions its solution can be written in the form

$$\varphi_k = hB_h\psi_k, \qquad \psi \in C_h[x_0, \bar{x}], \tag{7}$$

where, for every fixed h, B_h is a linear bounded operator with norm

$$\|B_h\| = \max_{0\le k\le N_h} \sum_{i=0}^{k-1} |\gamma_{ki}|,$$

where the γ_{ki} are completely determined by some fundamental system of solutions of the homogeneous equation corresponding to equation (6).

Analogously, the finite-difference equation

$$\sum_{i=0}^{m-1} \sum_{j=0}^{i} \alpha_j \varphi(x - ih) = h\psi(x) \tag{7'}$$

under zero initial conditions determines the function

$$\varphi(x) = h\widetilde{B}_h\psi(x), \qquad \psi \in C[x_0, \bar{x}], \tag{8'}$$

where the operator $\widetilde{B}_h$, for every fixed h, is linear and bounded, $\|\widetilde{B}_h\| = \|B_h\|$.

Let f range over the set of all continuous functions in $\bar{G}$; then the function ψ_h, defined by the relation

$$\psi_k = \sum_{i=0}^{n} \beta_i f_{k+l+1-i}, \qquad 0 \le k \le N_h = \left[\frac{\bar{x} - x_0}{h}\right], \tag{9}$$

ranges over the whole space $C_h[x_0, \bar{x}]$.

Denote by $K(C)$ the class of all functions ψ belonging to

$$\sum_{0<h<h_0} C_h[x_0, \bar{x}],$$

for which $\|\psi\| \le C$, $C > 0$, and by $R(C')$ the class

all continuously differentiable functions $g(x)$, $x^* \le x \le x_0$, for which $M_{g'} \le C'$, $C' > 0$, $g(x_0) = y_0$.*

The finite-difference process defined by equation (3) will be called uniformly convergent if it converges and if, whatever $C > 0$ and $C' > 0$ may be, there exists a constant $A_{CC'}$, depending only on C and C', such that the inequality

$$|\Delta y_k/h| < A_{CC'} \tag{10}$$

is satisfied uniformly with respect to h, $0 \leq h \leq h_0$, with respect to $k = 0, 1, \ldots, N_h$, with respect to $\psi \in K(C)$, and with respect to $g \in R(C')$; $\Delta y_k = y_{k+1} - y_k$ is found by means of equation (3) and the initial conditions (4). The equation

$$\sum_{i=0}^{m-1} \sum_{j=0}^{i} \alpha_j \lambda^{m-1-i} = 0 \tag{11}$$

will, as usual, be called the **characteristic equation of the difference equation** (6).

Theorem 2. *Suppose that the coefficients of equation (3) satisfy conditions (5). Then, for the uniform convergence of the finite-difference process defined by this equation, it is necessary and sufficient that the norm of the operator* B_h *be uniformly bounded with respect to* h*,* $0 < h \leq h_0$*, that the moduli of the simple roots of the characteristic equation (11) not exceed unity, and that the moduli of its multiple roots be less than unity.*

Let us consider some special cases.

A. For the finite-difference process defined by the equation

$$\alpha_0 y_k + \alpha_1 y_{k-1} = h(\beta_0 f_{k+1} + \cdots + \beta_n f_{k+1-n}),$$

the necessary conditions (5) take the form

$$\alpha_0 + \alpha_1 = 0, \qquad \alpha_0 = \beta_0 + \cdots + \beta_n \neq 0,$$

and are also sufficient conditions for convergence.

B. The finite-difference process generated by the equation

$$\alpha_0 y_k + \alpha_1 y_{k-1} + \alpha_2 y_{k-2} = h(\beta_0 f_{k+1} + \cdots + \beta_n f_{k+1-n})$$

converges uniformly only when the conditions

$$\alpha_0 + \alpha_1 + \alpha_2 = 0, \qquad 2\alpha_0 + \alpha_1 = \beta_0 + \cdots + \beta_n \neq 0,$$

$$-1 < -\frac{\alpha_0 + \alpha_1}{\alpha_1} < 1$$

are fulfilled.

3°. Suppose that $f(x, y)$ satisfies the Lipschitz condition in y with constant L. Then the integral

$$u = \int_{x_0}^{x} |\tilde{y}_h(\xi) - y(\xi)| \, d\xi$$

satisfies the inequality

$$du/dx \leq Lu + A + B(x - x_0), \tag{12}$$

where

$$A = h\rho(h)\left[C^{**}\sum_{i=0}^{m-1}\left|\sum_{j=0}^{i}\alpha_j\right| i + C^{*}(m-1)\right], \qquad B = C^{**}\varepsilon(h),$$

$$\rho(h) = \max_{x_0-(m-1)h\leq x\leq x_0}\left|\frac{\Delta g_{i-1}}{h} - f(x, \tilde{y}_h(x))\right|, \qquad \|\widetilde{B}_h\| \leq C^{**},$$

$$\varepsilon(h) = \max_{x_0\leq x\leq \bar{x}}|Q_h(x)|, \tag{13}$$

$$Q_h(x) = \sum_{i=0}^{n}\beta_i f_{k+1-i} - \sum_{j=0}^{m-1}\sum_{j=0}^{i}\alpha_j f(x - ih, \tilde{y}_h(x - ih)), \qquad x_{k-1} < x \leq x_k,$$

$$C^{*} = \max_{0\leq k<+\infty}\sum_{i=1}^{m-1}\sum_{j=1}^{m-1}|C_{ij}|\,|\zeta_i(k)|,$$

where $\zeta_1(k), \ldots, \zeta_{m-1}(k)$ denote the functions, arranged in a certain order,

$$\lambda_1^k,\ k\lambda_1^k, \ldots,\ k^{\omega_1-1}\lambda_1^k, \ldots, \lambda_s^k,\ k\lambda_s^k, \ldots,\ k^{\omega_s-1}\lambda_s^k,$$

constructed by means of the roots of equation (11) (λ_i has multiplicity ω_i; $\sum_{i=0}^{s}\omega_i = m-1$), and the constants C_{ij} do not depend on h. From (12) one obtains the estimate

$$|\tilde{y}_h(x) - y(x)| \leq Ae^{L(x-x_0)} + \frac{B}{L}\left[e^{L(x-x_0)} - 1\right]. \tag{14}$$

If $f(x, y)$ satisfies the Lipschitz condition

$$|f(x, y) - f(x', y')| \leq L_1|x - x'| + L_2|y - y'|$$

with constants L_1 and L_2, then estimate (14) can be given a more effective form:

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* By M_f and m_f, M_ψ and m_ψ, etc., we shall denote the upper and lower bounds of the moduli of the functions f, ψ, etc., in the region under consideration.

$$|\tilde{y}_h(x) - y(x)| \leq h \left\{ P e^{L_2(x-x_0)} + \frac{C^{**}S}{L_2} \left[e^{L_2(x-x_0)} - 1 \right] \right\}, \tag{15}$$

where

$$P = \rho(h) \left[C^* \sum_{i=0}^{m-1} \left| \sum_{j=0}^{i} \alpha_j \right| i + C^*(m-1) \right], \qquad \rho(h) \leq M_{g'} + M_f,$$

$$S = \left[L_1 + L_2 \left(C^* M_{g'} + C^{**} M_f \right) \sum_{i=0}^{n} |\beta_i| \right] \left[\sum_{i=0}^{n} |\beta_i| (|i-l| + 1) + \sum_{i=0}^{m-1} \left| \sum_{j=0}^{i} \alpha_j \right| i \right].$$

4°. Let, for the equation

$$y'(x) = f(x, y(x)), y(x - \tau(x)), \qquad m_\tau > 0, \tag{16}$$

where $f(x, y, z)$ is given and continuous in the closed domain $\overline{G}$ of the space (x, y, z), and $\tau(x)$ is also a given continuous function, it be required to find a solution satisfying the condition

$$y(x) = \varphi(x) \quad \text{for } \tilde{x} \leq x \leq x_0, \tag{17}$$

where

$$\tilde{x} = \min_{x_0 \leq x \leq X} [x - \tau(x)];$$

$\varphi(x)$ is a given continuous function.

The definitions, propositions, and estimates established in paragraphs $1^\circ, 2^\circ, 3^\circ$ extend, with slight modifications, to the finite-difference method for solving problem (16), (17), defined as follows. Consider the points x_i, $i = 0, \pm 1, \pm 2, \dots$, and use the finite-difference equation

$$\sum_{i=0}^{m} \alpha_i y_{k-i} = h \sum_{i=0}^{n} \beta_i f_{k+l-i} \tag{18}$$

of order $p - q$. Put

$$y_i = g(x_i), \qquad i = 0, -1, \dots, -(p-q-1), \tag{19}$$

where $g(x)$ is a continuously differentiable function defined on the interval $x^* \leq x \leq x_0$, $g(x_0) = \varphi(x_0)$; the step h must satisfy the conditions

$$0 < h \leq m_\tau, \qquad h(p-q-1) \leq x^* - x_0.$$

Next we shall solve equation (18) step by step under the initial conditions (19), putting $\tilde{y}_h(x) = \varphi(x)$, $\tilde{x} \leq x \leq x_0$,

$$f_j = f\left(x_j, y_j, \tilde{y}_h(x_j - \tau(x_j))\right),$$

and continue, step by step, the function $y = \tilde{y}_h^{\circ}(x)$, applying linear interpolation of the neighboring values y_j and y_{j+}. This will lead to the construction of an approximate solution of problem (16), (17).

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