MATHEMATICS

A. Ya. POVZNER and I. V. SUKHAREVSKII

1958

SovietRxiv

Abstract

Full Text

MATHEMATICS

A. Ya. POVZNER and I. V. SUKHAREVSKII

ON DISCONTINUITIES OF THE GREEN FUNCTION OF A MIXED PROBLEM FOR THE WAVE EQUATION AND ON SOME DIFFRACTION PROBLEMS

(Presented by Academician S. L. Sobolev on 5 VI 1958)

Consider the following mixed problem. In a two-dimensional domain D, bounded by a simple infinitely differentiable contour S (closed or else going off to infinity), it is required to find a solution of the equation

$$\Delta u = u_{tt},$$

satisfying the conditions*

$$u(0,x) = 0; \qquad u_t(0,x) = f(x); \qquad u|_S = 0.$$

The solution of this problem can be represented in the form

$$u(t,x) = u_0(t,x) + \int_D w(t,x,y) f(y)\, d\omega_y,$$

where $u_0(t,x)$ is the solution of the Cauchy problem in the whole space with the same initial conditions ($f(x)$ being continued by zero outside D). We shall be interested in the points $t = t_k(x,y)$ of discontinuity of the function w and of its derivatives with respect to t, and in the magnitudes of the corresponding jumps.

If one introduces the Green function

$$v(x,a,k) = \frac{1}{2\pi} H_0^{(1)}(k|x-a|) + \gamma(x,a,k)$$

of the Dirichlet problem for the equation

$$\Delta v + k^2 v = 0$$

in the domain D, then it is easy to establish that

$$\gamma(x, a, k) = \int_0^{+\infty} e^{ikt} w(t, x, a)\, dt.$$

This points to a close formal connection between our problem and the fundamental problem of the theory of diffraction for short waves—the problem of finding the asymptotics of the function $\gamma(x, a, k)$ for large k: integration by parts in the integral representing γ expresses the desired asymptotics through the dis-

---

* x, y are the radius vectors of points in $\overline{D}$; a is a point of D fixed in what follows. If $s \in S$, then ξ is the arc coordinate of the point $s = s(\xi)$.

discontinuities of the function w. If we seek the function γ in the form of a double-layer potential

$$\gamma(x, a, k) = \frac{1}{2\pi} \int_S g(s, k) \frac{\partial}{\partial n_s} H_0^{(1)}(k|x - s|)\, d\xi, \tag{1}$$

then we obtain for $g(s, k)$ the integral equation

$$g(x, k) - \int_S h(x, s, k) g(s, k)\, ds = h_0(x, a, k) \qquad (x \in S), \tag{2}$$

where

$$h_0(x, a, k) = -\frac{1}{\pi} \int_0^{\infty} \frac{e^{ikt} \sigma(t - |x - a|)}{\sqrt{t^2 - |x - a|^2}}\, dt;$$

$$h(x, s, k) = \frac{ik}{\pi} \frac{\partial}{\partial n_s} \ln|x - s| \cdot \int_0^{\infty} \frac{e^{ikt}\, t\sigma(t - |x - s|)}{\sqrt{t^2 - |x - s|^2}}\, dt$$

($\sigma(t) = 1$ for $t > 0$; $\sigma(t) = 0$ for $t < 0$; n_s is the unit vector of the exterior normal, $\operatorname{Im} k > 0$).

Put

$$g_0(x, k) = h_0(x, a, k),$$

$$g_{\mu+1}(x, k) = \int_S h(x, s, k) g_\mu(s, k)\, ds \qquad (\mu = 0, 1, 2, \ldots),$$

$$\sum_{\mu=0}^{n} g_\mu(x,k) = g_n^{(1)}(x,k).$$

Represent g in the form

$$g(x,k) = g_n^{(1)}(x,k) + g_n^{(2)}(x,k).$$

Then $g_n^{(2)}$ satisfies the equation

$$g_n^{(2)}(x,k) - \int_S h(x,s,k) g_n^{(2)}(s,k)\,ds = g_{n+1}(x,k), \tag{2'}$$

and the function

$$\gamma(x,a,k) = \gamma_n^{(1)}(x,a,k) + \gamma_n^{(2)}(x,a,k),$$

where $\gamma_n^{(j)}$ is a potential of type (1) with density $g_n^{(j)}$. Further, representing $\gamma_n^{(j)}$ in the form

$$\gamma_n^{(j)} = \int_0^\infty e^{ikt} w_n^{(j)}(t,x,a)\,dt,$$

we obtain:

$$w(t,x,a) = w_n^{(1)}(t,x,a) + w_n^{(2)}(t,x,a).$$

The principal result of the present note is the following theorem:

Theorem 1. If the point a is such that from it one cannot draw a single tangent to S* (the contour S is completely "illuminated" from the point a), then for any prescribed $T > 0$ and natural number l there exists such an n that

* This property is possessed, for example, by all interior points of convex domains.

$w_n^{(2)}(t,x,a)$ will be continuous, together with all mixed derivatives with respect to t and the coordinates of the vector x, up to order l inclusive in the domain $(0 \le t \le T;\ x \in \overline{D})$, and

$$\frac{\partial^p w_n^{(2)}(0,x,a)}{\partial t^p} = 0 \qquad (p = 0,1,2,\dots,l).$$

Taking into account that $w_n^{(1)}$ is computed directly from $g_n^{(1)}$, we thereby obtain an effective method for finding the discontinuities of the function w.

Let us represent $g_{n+1}(s,k)$ in the form

$$g_{n+1}(s,k) = \int_0^\infty e^{ikt} w_{n+1}(t,s,a)\,dt \qquad (s \in S).$$

Suppose that, for sufficiently large n, the function $w_{n+1}(t,s,a)$ has, in the given interval $[0;T]$, an arbitrarily large number of mixed derivatives with respect to t and ξ equal to zero for $t=0$. Since the function $w_n^{(2)}$ is the solution of the mixed problem

$$\Delta u = u_{tt}; \qquad u(0,x) = u_t(0,x) = 0; \qquad u(t,s) = w_{n+1}(t,s,a) \qquad (s \in S),$$

it follows, by virtue of a known existence theorem, that the function $w_n^{(2)}(t,x,a)$, for sufficiently large n, has the properties indicated in Theorem 1. In the present paper, in fact, it is proved that the above assumption concerning w_{n+1} does indeed hold.

From the explicit representation of the function $w_n^{(1)}$, in particular, it follows:

Theorem 2. The discontinuities of the function $w(t,x,a)$ and of its derivatives with respect to t are located at the point $t = |x-a|$ and at the points

$$t = |x-\bar{s}_1| + |\bar{s}_1 - \bar{s}_2| + \cdots + |\bar{s}_m - a| \qquad (m = 1, 2, \ldots),$$

where $\bar{s}_i$ are all points of the contour S having the property that adjacent sides of the polygons with endpoints x and a and with vertices at $\bar{s}_1, \ldots, \bar{s}_m$ make equal angles with the normal to S at the common vertex.*

The proof of Theorem 1 is based on a number of auxiliary propositions, the principal ones of which are given below.

Denote by $\alpha = \alpha(a,T)$ the lower bound of the lengths of the links of extremal polygons having one end on S and perimeter not exceeding T. It can be shown that $\alpha > 0$.** Introduce functions

$$u_1^{(m)}(s_1,s_2,\varepsilon), \qquad u_2^{(m)}(s_1,s_2,\delta), \qquad u_3^{(m)}(s_1,s_2,\delta,\varepsilon) \quad (s_i = s(\xi_i)),$$

having continuous mixed derivatives with respect to ξ_1, ξ_2 up to order m inclusive and defined so that $u_1^{(m)} = 0$ for $|s_1 - s_2| \geq \varepsilon$, $u_1^{(m)} = 1$ for $|s_1 - s_2| \leq \varepsilon/2$, and is positive at the remaining points; $u_2^{(m)} = 0$ for $|s_1 - s_2| \leq \alpha - \delta$; $u_2^{(m)} = 1$ for $|s_1 - s_2| \geq \alpha - \delta/2$, and is positive at the remaining points ($\varepsilon < \alpha/2 - \delta$); $u_3^{(m)} = 1 - u_1^{(m)} - u_2^{(m)}$. Introduce the iterated kernels

$$h^p(s,a,k) = \int_S \cdots \int_S h(s,s_p,k) \cdots h(s_2,s_1,k) h_0(s_1,a,k)\, ds_1 \cdots ds_p$$

and represent them in the form

$$h^{(p)} = \sum_{j_0,\ldots,j_1=1}^{3} h^{(p)}_{j_p \cdots j_1},$$

* Such polygons will be called **extremal** in what follows.

** We owe the proof of this proposition to A. V. Pogorelov.

where

$$h^{(p)}_{j_p \cdots j_1}(s,a,k) = \int_S \cdots \int_S h(s,s_p,k)\, u^{(m)}_{j_p}\left(s, s_p, \delta, \frac{\varepsilon}{2^p}\right) \times$$

$$\times\, h(s_p,s_{p-1},k)\, u^{(m)}_{j_{p-1}}\left(s_p, s_{p-1}, \delta, \frac{\varepsilon}{2^{p-1}}\right) \cdots h_0(s_1,a,k)\, ds_1, \ldots, ds_p =$$

$$= \int_0^\infty e^{ikt}\, \tilde{h}^{(p)}_{j_p \cdots j_1}(t,s)\, dt.$$

Theorem 3. For each given l and n there exists an M such that, for $m \geq M$, the functions $\sum \tilde{h}^{(n+1)}_{j_{n+1} \cdots j_1}(t,s)$, where the summation is taken over all combinations $j_{n+1}, \ldots, j_1$ containing at least one index j equal to 3, have mixed derivatives with respect to t and ξ $(0 \leq t \leq T;\ s \in S)$ up to order l inclusive, equal to zero at $t = 0$.

Theorem 4. For any prescribed l, one can indicate such n and $M(n)$ that, if $m \geq M(n)$, then the functions $\tilde{h}^{(n+1)}_{j_{n+1} \cdots j_1}(t,s)$, in which none of the indices is equal to 3, have mixed derivatives with respect to t and ξ $(0 \leq t \leq T;\ s \in S)$ up to order l inclusive, equal to zero at $t = 0$.

Theorems analogous to Theorems 1 and 2 can be proved by the same method in the case of a space of any number of dimensions and under a boundary condition of the general form $\partial u/\partial n + q(s)u = 0$. Moreover, these same results can also be extended to the case of Maxwell' s equations, including problems connected with diffraction by infinitely thin screens of a prescribed form.

Kharkov State University
named after A. M. Gorky

Received
4 VI 1958

Note: Figure translations are in progress. See original paper for figures.