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Abstract

Full Text

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THE METHOD OF AVERAGING CANONICAL EQUATIONS CONTAINING A "QUASI-CYCLIC" ANGULAR COORDINATE

(Presented by Academician N. N. Bogolyubov, December 2, 1957)

Let us consider a dynamical system whose state is determined by r variables $q_1, q_2, \dots, q_r$ and an angular variable φ. Suppose that the motion of the system under consideration is described by a canonical system of equations of the form

$$\frac{dq_k}{dt} = \frac{\partial H}{\partial p_k}, \tag{1a}$$

$$(k = 1, 2, \dots, r)$$

$$\frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k}; \tag{1 }$$

$$\frac{d\varphi}{dt} = \frac{\partial H}{\partial p_{r+1}}, \qquad \frac{dp_{r+1}}{dt} = -\mu \frac{\partial H_1}{\partial \varphi}, \tag{2}$$

where

$$H = H_0(q_1, \dots, q_r,\ p_1, \dots, p_r,\ p_{r+1}) +$$

$$+\mu H_1(q_1, \dots, q_r, \varphi,\ p_1, \dots, p_{r+1}) + \mu^2 \dots; \tag{3}$$

$$H_0 = \frac{1}{2}\sum_{i=1}^{r}\sum_{k=1}^{r} a_{ik} q_{iq} k + \sum_{i=1}^{r}\sum_{k=1}^{r+1} b_{ik} q_{ip} k + \frac{1}{2}\sum_{k=1}^{r+1} c_{kp} k^2; \tag{4}$$

μ is a small parameter.

Systems of this type occur in the dynamics of turbogenerator rotors; moreover, the unperturbed Hamiltonian H_0 is readily reduced to the form (4) by a suitable choice of generalized coordinates.

In equations (2) the derivative of the momentum coordinate p_{r+1}, corresponding to the angular variable φ, is proportional to the small parameter; therefore, according to N. N. Bogolyubov' s perturbation theory ([1]), it is a slowly varying function of time, and the angular variable may be called "quasi-cyclic" (i.e., almost cyclic). From physical considerations it follows that the coordinates $q_1, q_2, \dots, q_r$ are periodic functions of the angle of rotation φ with period 2π.

Following the idea of the asymptotic methods of N. M. Krylov and N. N. Bogolyubov ([2]), we assume for system (1), in the first approximation,

$$q_k^{(1)} = a_k \cos(\varphi + \psi_k), \tag{5}$$

where a_k and ψ_k are regarded as slowly varying functions of time which, over the course of one period, may be considered constant; as a result we have

$$\dot{q}_k = -a_k \dot{\varphi} \sin(\varphi + \psi_k). \tag{6}$$

Solving equation (1a) with respect to the momentum coordinates $p_1, p_2, \dots, p_r$, we obtain

$$p_k = -\frac{1}{c_k} a_k \dot{\varphi} \sin(\varphi + \psi_k) - \frac{1}{c_k} \sum_{i=1}^{r} b_{ik} a_i \cos(\varphi + \psi_i) - \frac{\mu}{c_k} \left(\frac{\partial H_1}{\partial p_k} \right)^{(1)}. \tag{7}$$

Considering expressions (5) and (7) as formulas for a transformation of variables and differentiating them, taking into account the dependence of a_k and ψ_k on time, we obtain

$$\frac{da_k}{dt} \cos\theta_k - a_k \frac{d\psi_k}{dt} \sin\theta_k = 0, \tag{8}$$

$$\frac{da_k}{dt} \sin\theta_k + a_k \frac{d\psi_k}{dt} \cos\theta_k =$$

$$= \frac{c_k}{\dot{\varphi}} \left(\frac{\partial H}{\partial q_k} \right)^{(1)} - \frac{a_k \ddot{\varphi}}{\dot{\varphi}} \sin\theta_k - \frac{\mu}{\dot{\varphi}} \frac{d}{dt} \left(\frac{\partial H_1}{\partial p_k} \right)^{(1)} - a_k \dot{\varphi} \cos\theta_k -$$

$$- \frac{1}{\dot{\varphi}} \sum_{i=1}^{r} \left(\frac{db_{ik}}{dt} a_i \cos\theta_i - b_{ik} a_i \dot{\varphi} \sin\theta_i \right) =$$

$$= F_k(a_1, \dots, a_r, \varphi + \psi_1, \dots, \varphi + \psi_r), \tag{9}$$

where $\theta_k = \varphi + \psi_k$. In the expressions for the derivatives $\partial H / \partial q_k$ and $\partial H_1 / \partial p_k$ in equations (9), the values of q_k and p_k according to formulas (5) and (7) must be substituted.

Multiplying equation (8) successively by $\cos\theta_k$ and $\sin\theta_k$, and equation (9), respectively, by $\sin\theta_k$ and $\cos\theta_k$, we obtain, as the result of addition and subtraction, the following system of equations

$$\frac{da_k}{dt} = F_k(a_1, a_2, \ldots, a_r, \varphi + \psi_1, \ldots, \varphi + \psi_r)\sin\theta_k,$$

$$\frac{d\psi_k}{dt} = \frac{1}{a_k} F_k(a_1, a_2, \ldots, a_r, \varphi + \psi_1, \ldots, \varphi + \psi_r)\cos\theta_k. \tag{10}$$

In order to eliminate the "quasicyclic" variable from the right-hand sides of equations (10), let us average them over $\varphi + \psi_k$ over a time equal to one period; we obtain

$$\frac{da_k}{dt} = -\frac{a_k\ddot{\varphi}}{2\dot{\varphi}} - \frac{1}{2\dot{\varphi}}\sum_{i=1}^{r}\left[\frac{db_{ik}}{dt}a_i\sin(\psi_i - \psi_k) - b_{ik}a_i\dot{\varphi}\cos(\psi_i - \psi_k)\right] +$$

$$+\frac{1}{2\pi\dot{\varphi}}\int_0^{2\pi}\left[c_k\left(\frac{\partial H}{\partial q_k}\right)^{(1)} - \mu\frac{d}{dt}\left(\frac{\partial H_1}{\partial p_k}\right)^{(1)}\right]\sin\theta_k\,d\theta_k = \Phi_k(a_1, \ldots, a_r, \psi_1, \ldots, \psi_r), \tag{11}$$

$$\frac{d\psi_k}{dt} = -\frac{\dot{\varphi}}{2} - \frac{1}{2\dot{\varphi}a_k}\sum_{i=1}^{r}\left[\frac{db_{ik}}{dt}a_i\cos(\psi_i - \psi_k) - b_{ik}a_i\dot{\varphi}\sin(\psi_i - \psi_k)\right] +$$

$$+\frac{1}{2\pi\dot{\varphi}a_k}\int_0^{2\pi}\left[c_k\left(\frac{\partial H}{\partial q_k}\right)^{(1)} - \mu\frac{d}{dt}\left(\frac{\partial H_1}{\partial p_k}\right)^{(1)}\right]\cos\theta_k\,d\theta_k = \Psi_k(a_1, \ldots, a_r, \psi_1, \ldots, \psi_r).$$

Equating to zero the right-hand sides of equations (11) and averaging equations (2) for the "quasicyclic" coordinate, we obtain equations for determining

parameters of the stationary motion:

$$\Phi_k(a_1, \ldots, a_r, \psi_1, \ldots, \psi_r) = 0,$$

$$\Psi(a_1, \ldots, a_r, \psi_1, \ldots, \psi_r) = 0; \tag{12}$$

$$\frac{d^2\varphi}{dt^2} = \frac{\mu}{2\pi}\int_0^{2\pi}\left[-c_k\left(\frac{\partial H_1}{\partial\varphi}\right)^{(1)} + \frac{d}{dt}\left(\frac{\partial H}{\partial p_{n+1}}\right)^{(1)}\right]d\theta_k = 0. \tag{13}$$

Having found the values a_k^0 and ψ_k^0 from equations (12) and (13), and the expressions $\bar{q}_k^{(1)}, \bar{p}_k^{(1)}$, which characterize the values of the generalized and momentum coordinates in stationary motion, we investigate its stability.

According to A. M. Lyapunov' s theory of stability [3], a sufficient condition for the stability of the motion of a canonical system is the sign-definiteness of the quadratic form

$$H_2 = \frac{1}{2} \sum_{i=1}^{r+1} \sum_{k=1}^{r+1} \left[\left(\frac{\overline{\partial^2 H}}{\partial q_i \, \partial q_k} \right) \xi_i \xi_k + 2 \left(\frac{\overline{\partial^2 H}}{\partial q_i \, \partial p_k} \right) \xi_i \eta_k + \left(\frac{\overline{\partial^2 H}}{\partial p_i \, \partial p_k} \right) \eta_i \eta_k \right] \tag{14}$$

$$(i, k = 1, 2, \dots, r+1),$$

formed from the lowest quadratic terms of the expansion of the Hamiltonian in a Taylor series in powers of the perturbations ξ_i, η_i of the generalized and momentum coordinates.

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CITED LITERATURE

[1] N. N. Bogolyubov, *On certain statistical methods in mathematical physics*, Kiev, 1945. [2] N. M. Krylov, N. N. Bogolyubov, *Introduction to nonlinear mechanics*, Kiev, 1937. [3] A. M. Lyapunov, *The general problem of the stability of motion*, 1950.

Note: Figure translations are in progress. See original paper for figures.