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Abstract

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MATHEMATICAL PHYSICS

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ANALYTIC PROPERTIES OF THE AMPLITUDE OF A PROCESS WITH A VARIABLE NUMBER OF PARTICLES

(Presented by Academician N. N. Bogolyubov, 17 II 1958)

At present, the method of dispersion relations is widely used in quantum field theory. Although it does not give complete information about one or another specific process, it nevertheless leads to certain exact relations between observable quantities. Despite the fact that the method of dispersion relations was proposed in a number of works as early as 1954, until recently a number of unclear questions remained in the justification of dispersion relations. Only at the end of 1956 did N. N. Bogolyubov succeed in overcoming the great difficulties present here and in constructing a rigorous derivation of dispersion relations for meson-nucleon scattering processes. In doing so, N. N. Bogolyubov had to carry out a deep analysis of the basic physical assumptions of the derivation and to prove a number of delicate theorems from the field bordering the theory of functions of complex variables and functional analysis. In the note (1) dispersion relations were considered for processes with a variable number of particles.

In the present note (using the example of the double Compton effect on a nucleon), the method of N. N. Bogolyubov (2) is applied to the proof of these relations in the case of absence of an unobservable energy region.

The Fourier transforms of the retarded and advanced matrix elements of the double Compton effect can be written in the form (1):

$$\begin{aligned}
 T_{\alpha,\omega}^{\text{ret}}(E, \mathbf{Q}, \Delta) &= - \int e^{iQx+i\Delta y} F_{\alpha,\omega}^{\text{ret}}(x, y) dx dy, \\
 T_{\alpha,\omega}^{\text{adv}}(E, \mathbf{Q}, \Delta) &= \int e^{iQx+i\Delta y} F_{\alpha,\omega}^{\text{adv}}(x, y) dx dy.
 \end{aligned}
 \tag{1}$$

According to the causality condition, the integration in (1) with respect to the variable x is carried out in the region

$$x_0 \geq |\mathbf{x}|, \quad x_0 > 0
 \tag{2}$$

for the retarded amplitude and in the region

Fig. 1

Figure 1: Fig. 1

$$|x_0| \geq |\mathbf{x}|, \quad x_0 < 0 \quad (3)$$

for the advanced amplitude. Usually E and \mathbf{Q} are related by the following condition

$$E^2 = \mathbf{Q}^2 + m_Q^2. \quad (4)$$

However, at this stage we shall regard E and \mathbf{Q} as independent variables, while the vectors $\bar{\Delta}$ and \mathbf{p} are considered fixed. The functions $T_{\alpha,\omega}^{\text{ret}}$ and $T_{\alpha,\omega}^{\text{adv}}$ are defined everywhere on the real axis for $|E| > E_p$ (the threshold of the process).

Consider the function

$$T(E, \mathbf{Q}, \Delta) = T^{\text{ret}}(E, \mathbf{Q}, \Delta) - T^{\text{adv}}(E, \mathbf{Q}, \Delta). \quad (5)$$

This function has the following energy spectrum (Fig. 1),

Fig. 1

where the poles $\pm E_1$ and $\pm E_2$ are determined from the equations:

$$\begin{aligned} 4E^2(\mathbf{p}^2 + M^2) - [Q^2 + \Delta^2 + 2\mathbf{p}\mathbf{Q}]^2 &= 0, \\ 4E^2(\mathbf{p}^2 + M^2) - [Q^2 + \mathbf{p}\mathbf{Q}]^2 &= 0, \end{aligned} \quad (6)$$

and the boundary of the continuous spectrum is

$$E_q = \frac{1}{2}(M + \mu - p_0). \quad (7)$$

Let us construct a function $\tilde{T}(E, \mathbf{Q}, \Delta)$ that vanishes in the region $|E| < E_q$. To this end, instead of the functions $F^{\text{ret}}(x, y)$ and $F^{\text{adv}}(x, y)$, we introduce the functions $\tilde{F}^{\text{ret}}(x, y)$, $\tilde{F}^{\text{adv}}(x, y)$ in the following way:

$$\tilde{F}_{\text{adv}}^{\text{ret}}(x, y) = \left[4(\mathbf{p}^2 + M^2) \frac{\partial^2}{\partial x_0^2} + \left[\square_x^2 + \square_y^2 + 2i \left(p_i \frac{\partial}{\partial x_j} \right) \right]^2 \right] \left[(\mathbf{p}^2 + M^2) \frac{\partial^2}{\partial x_0^2} + \left[\square_x^2 + i \left(p_j \frac{\partial}{\partial x_j} \right) \right]^2 \right] F_{\text{adv}}^{\text{ret}}(x, y) \quad (8)$$

or, in the momentum representation,

$$\tilde{T}_{\text{adv}}^{\text{ret}}(E, \mathbf{Q}, \Delta) = \{[Q^2 + \Delta^2 + 2\mathbf{p}\mathbf{Q}]^2 - 4E^2(\mathbf{p}^2 + M^2)\} \{[Q^2 + \mathbf{p}\mathbf{Q}]^2 - E^2(\mathbf{p}^2 + M^2)\} T_{\text{adv}}^{\text{ret}}(E, \mathbf{Q}, \Delta). \quad (9)$$

The polynomial in (9) is chosen so as to eliminate the δ -singularities of the function $T(E, \mathbf{Q}, \Delta)$ at the points $\pm E_1, \pm E_2$. It is obvious that

$$\tilde{T}(E, \mathbf{Q}, \Delta) = \tilde{T}^{\text{ret}}(E, \mathbf{Q}, \Delta) - \tilde{T}^{\text{adv}}(E, \mathbf{Q}, \Delta) = 0, \quad (10)$$

if $|E| < E_q$, $\text{Im } E = 0$. We introduce the functions $\Phi^r(E, \mathbf{e}, \mathbf{p}, \vec{\Delta}, \rho)$ and $\Phi^a(E, \mathbf{e}, \mathbf{p}, \vec{\Delta}, \rho)$ as follows:

$$\Phi^{r,a}(E, \mathbf{e}, \mathbf{p}, \Delta, \rho) = \mp S_{\pm} \int \tilde{F}^{r,a}(x, y) \exp \left[iEx_0 - i\mathbf{e}\mathbf{x} \sqrt{E^2 - E_p^2} - i\mathbf{x}\mathbf{p}\eta + i\Delta y - \rho x^2 \right] dx dy, \quad (11)$$

where $\rho > 0$,

$$\eta = \frac{1}{2} \left(1 - \frac{m_Q^2}{\mathbf{p}^2} \right);$$

S_{\pm} is the operation of symmetrization or antisymmetrization with respect to \mathbf{e} , necessary for removing the double-valuedness of the square root $\sqrt{E^2 - E_p^2}$. The functions Φ^a, Φ^r are analytic in the upper (lower) half-plane of the energy E , respectively. On the basis of (10) we have:

$$\Phi^r(E, \mathbf{e}, \mathbf{p}, \vec{\Delta}, \rho) = \Phi^a(E, \mathbf{e}, \mathbf{p}, \vec{\Delta}, \rho) \quad \text{for } |E| < E_q, \quad \text{Im } E = 0. \quad (12)$$

Consequently, there exists a function $\Phi(E, \mathbf{e}, \mathbf{p}, \vec{\Delta}, \rho)$

$$\Phi(E, \mathbf{e}, \mathbf{p}, \vec{\Delta}, \rho) = \begin{cases} \Phi^r(E, \mathbf{e}, \mathbf{p}, \vec{\Delta}, \rho), & \text{for } \text{Im } E > 0, \\ \Phi^a(E, \mathbf{e}, \mathbf{p}, \vec{\Delta}, \rho), & \text{for } \text{Im } E < 0. \end{cases} \quad (13)$$

analytic in the entire energy plane except for the lines of the cuts:

$$-\infty < \text{Re } E < -E_q; \quad E_q < \text{Re } E < \infty; \quad \text{Im } E = 0; \quad (14)$$

here the values of Φ on the upper edges of the cuts are equal to Φ^r , and on the lower edges to Φ^a . If the function $\Phi(E, \mathbf{e}, \mathbf{p}, \vec{\Delta}, \rho)$, as $E \rightarrow \infty$, has a degree of growth not higher than $n + 8$, then to the function

Fig. 2

Figure 2: Fig. 2

$$(E - E_0)^{-n-9} \Phi(E, \mathbf{e}, \mathbf{p}, \vec{\Delta}, \rho), \quad (15)$$

where E_0 is a real parameter, $|E_0| < E_q$, one may apply the integral theorem with the contour of integration shown in Fig. 2. Letting the radius of the large circle tend to infinity, and the radii of the small semicircles to zero, we obtain

Fig. 2

$$\begin{aligned} \Phi(E, \mathbf{e}, \mathbf{p}, \vec{\Delta}, \rho) &= \frac{(E - E_0)^{n+2}}{2\pi i} \int_{-\infty}^{-E_q} \frac{\Phi(E' + i0, \mathbf{e}, \mathbf{p}, \vec{\Delta}, \rho) - \Phi(E' - i0, \mathbf{e}, \mathbf{p}, \vec{\Delta}, \rho)}{(E' - E)(E' - E_0)^{n+9}} dE' \\ &+ \frac{(E - E_0)^{n+9}}{2\pi i} \int_{E_q}^{\infty} \frac{\Phi(E' + i0, \mathbf{e}, \mathbf{p}, \vec{\Delta}, \rho) - \Phi(E' - i0, \mathbf{e}, \mathbf{p}, \vec{\Delta}, \rho)}{(E' - E)(E' - E_0)^{n+9}} dE' = P_{n+9}(E). \end{aligned} \quad (16)$$

It should be emphasized that in the case considered by us (there is no unobservable energy region) $E_q > E_p$, and the integration in (16) is carried out over the observable region. But since the functions Φ^r and Φ^a are defined in the observable region also for $\rho = 0$, we may put $\rho = 0$ in the integrals. It is then quite obvious that the right-hand side of expression (16) will be an analytic function in the entire plane of the complex variable E , except for the lines of the cuts (14). But from expression (16) it follows that, for $\rho = 0$, the right-hand side determines the function $\Phi(E, \mathbf{e}, \mathbf{p}, \vec{\Delta}, 0)$, analytic in the entire E -plane except for the lines of the cuts (14). But the function $\Phi(E, \mathbf{e}, \mathbf{p}, \Delta, 0)$ differs from the function

$$S_{\pm} G(E, \mathbf{e}, \mathbf{p}, \vec{\Delta}) = \begin{cases} S_{\pm} T^{\text{ret}}(E, \mathbf{e}, \mathbf{p}, \vec{\Delta}), & \text{for } \text{Im } E > 0, \\ S_{\pm} T^{\text{adv}}(E, \mathbf{e}, \mathbf{p}, \vec{\Delta}), & \text{for } \text{Im } E < 0 \end{cases} \quad (17)$$

only by factors that vanish at the points

$$E = \pm E_1, \quad E = \pm E_2. \quad (18)$$

Therefore the function $S_{\pm} G(E, \mathbf{e}, \mathbf{p}, \vec{\Delta})$ will be analytic in the entire plane of the complex variable E , with first-order poles at the points $E = \pm E_1$, $E = \pm E_2$, and with the cut lines (14). The function $S_{\pm} G(E, \mathbf{e}, \mathbf{p}, \vec{\Delta})$ has a degree of growth not higher than n . Consequently, to the function

Fig. 3

Figure 3: Fig. 3

$$(E - E_0)^{-n-1} S_{\pm} G(E, \mathbf{e}, \mathbf{p}, \vec{\Delta}) \quad (19)$$

one can apply Cauchy's integral theorem with the contour of integration shown in Fig. 3.

Fig. 3

Letting the radius of the large circle tend to infinity, and the radius of the small semicircles to zero, we obtain

$$S_{\pm} D_{\alpha, \omega}(E) = \frac{(E - E_0)^{n+1}}{\pi} P \int_{|E'| > E_q} dE' \frac{S_{\pm} A_{\alpha, \omega}(E')}{(E' - E)(E' - E_0)^{n+1}} +$$

$$+ \sum_{i=1,2} \left[\left(\frac{E_0 - E}{E_0 - E_i} \right)^{n+1} \frac{S_{\pm} R_i(\mathbf{p}, \Delta, \mathbf{e})}{E_i + E} + \left(\frac{E_0 - E}{E_0 - E_i} \right)^{n+1} \frac{S_{\pm} \Omega_i(\mathbf{p}, \Delta, \mathbf{e})}{E_i - E} \right] + P_n(E). \quad (20)$$

In deriving (20) it was taken into account that

$$\lim_{\varepsilon \rightarrow 0} S_{\pm} G(E + i\varepsilon) = S_{\pm} T^{\text{ret}}(E), \quad \lim_{\varepsilon \rightarrow 0} S_{\pm} G(E - i\varepsilon) = S_{\pm} T^{\text{adv}}(E),$$

$$D_{\alpha, \omega}(E) = \frac{1}{2} (T_{\alpha, \omega}^{\text{ret}}(E) + T_{\alpha, \omega}^{\text{adv}}(E)),$$

$$A_{\alpha, \omega}(E) = \frac{1}{2i} (T_{\alpha, \omega}^{\text{ret}}(E) - T_{\alpha, \omega}^{\text{adv}}(E)).$$

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REFERENCES

1. A. A. Logunov, *DAN*, **120**, No. 3 (1958).
2. N. N. Bogolyubov, D. V. Shirkov, *Introduction to the Theory of Quantized Fields*, Moscow, 1957.

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