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## Abstract

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## MATHEMATICS

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# ON THE CONVERGENCE OF METHODS OF GALERKIN TYPE

*(Presented by Academician S. L. Sobolev on 14 I 1958)*

In the present article one general scheme for the application of methods of Galerkin type is considered, and simple conditions for the convergence of these methods are given.

Let $X$ and $Y$ be Hilbert spaces, and let $\{\varphi_i\}$ be a complete orthonormal system in $X$. Consider the equation

$$Gu \equiv Au + \lambda K u = v, \tag{1}$$

where the operators $A$ and $K$ are linear and act from $X$ into $Y$, the operator $A$ has a bounded inverse $A^{-1}$, and $v \in Y$. Let $\{\psi_i\}$ be some system of elements of $Y$ satisfying, for every $n$, the condition

$$\begin{vmatrix} (A\varphi_1, \psi_1) & \cdots & (A\varphi_1, \psi_n) \\ \cdots & \cdots & \cdots \\ (A\varphi_n, \psi_1) & \cdots & (A\varphi_n, \psi_n) \end{vmatrix} \neq 0. \tag{2}$$

We seek a solution of equation (1) in the form

$$u_n = \sum_{k=1}^{n} a_k \varphi_k,$$

where the unknown constants $a_1, \dots, a_n$ are determined from the conditions

$$(Au_n + \lambda K u_n - v, \psi_m) = 0 \qquad (m = 1, \dots, n)$$

or

$$\sum_{k=1}^{n}\{(A\varphi_k,\psi_m)+\lambda(K\varphi_k,\psi_m)\}a_k=(v,\psi_m)\qquad(m=1,\dots,n).\tag{3}$$

This is the Galerkin–Petrov method ([1]).

Following L. V. Kantorovich ([2]) (see also ([3])), introduce the spaces $X'$, $Y'$, $\bar{X}$, $\bar{Y}$ and the operators $\varphi_0,\psi_0,\psi$. As the space $X'$ we take the set of elements of the form $x'=\sum_{k=1}^{n}\alpha_k\varphi_k$, and as the space $Y'$ the aggregate of elements of the form $y'=Ax'=\sum_{k=1}^{n}\alpha_k A\varphi_k$. Put $\bar{X}=\bar{Y}=R^n$, and we shall assume

$$\varphi_0 x'=\bar{x}(\alpha_1,\dots,\alpha_n),\qquad \psi_0 y'=\bar{y}\{(y',\psi_1),\dots,(y',\psi_n)\}.$$

The operator $\psi$ acts from $Y$ into $\bar{Y}$, where $\psi=\psi_0$ on $Y'$:

$$\psi y=\bar{y}\{(y,\psi_1),\dots,(y,\psi_n)\}.$$

Note that the operator $\varphi_0$ has an inverse $\varphi_0^{-1}$,

$$\varphi_0^{-1}\bar{x}=\sum_{k=1}^{n}\alpha_k\varphi_k.$$

Put

$$\psi_0^{-1}\bar{y}=\sum_{k=1}^{n}\alpha_k A\varphi_k,$$

where the coefficients $\alpha_1,\dots,\alpha_n$ are determined from the system

$$\sum_{k=1}^{n}(A\varphi_k,\psi_m)\alpha_k=(y,\psi_m)\qquad(m=1,\dots,n).$$

By virtue of condition (2), the latter system is solvable. We define the norm for the space $\overline{Y}$ by the equality $\|\overline{y}\|=\|\psi_0^{-1}\overline{y}\|$. In this case we shall have

$$\|\varphi_0\|=\|\varphi_0^{-1}\|=\|\psi_0\|=\|\psi_0^{-1}\|=\|\psi\|=1.$$

Introduce the notation $\overline{A}=\psi A\varphi_0^{-1}$, $\overline{K}=\psi K\varphi_0^{-1}$, $\overline{v}=\psi v$. We now write system (3) in the form

$$\overline{A}u+\lambda\overline{K}u=\overline{v}.\tag{4}$$

We impose a smoothness condition on the operator $K$: for every element $x \in K$ there is an element $y' \in Y'$ such that the inequality

$$\|Kx - y'\| \leqslant \varepsilon_n \|x\|, \qquad \varepsilon_n \to 0 \quad \text{as } n \to \infty. \tag{5}$$

holds.

Condition (5) may be replaced by another condition:

$$\|A^{-1}Kx - x'\| \leqslant \varepsilon_n' \|x\|, \qquad \varepsilon_n' \to 0 \quad \text{as } n \to \infty. \tag{6}$$

Let us also note that the condition $\|\overline{K}\varphi_0 x' - \psi K x'\| \leqslant \mu \|x'\|$ is satisfied trivially with $\mu = 0$.

We can now apply the theorem of L. V. Kantorovich ([2]), from which it follows:

**Theorem 1.** *If the operator $G$ has an inverse and conditions (2) and (5) or (6) are fulfilled, then, beginning with some $n$, equations (3) are solvable and $u_n$ converges strongly as $n \to \infty$ to the solution $u$ of equation (1).*

From Theorem 1 there follows an important theorem for applications.

**Theorem 2.** *If the operator $G$ has an inverse, condition (2) is fulfilled, and one of the following two conditions holds: 1) the operator $K$ is completely continuous; 2) the operator $A^{-1}K$ is completely continuous, then the assertion of Theorem 1 is valid.*

Let us clarify when, in particular, condition (2) is fulfilled. First of all, it is fulfilled if $A = E$ and $\psi_i = \varphi_i$. Theorem 2 in this case establishes the convergence of the ordinary Galerkin method for equations with a completely continuous operator. If the operator $A$ is positive-definite and $\psi_i = \varphi_i$, then condition (2) is also fulfilled, and Theorem 2 gives the result of S. G. Mikhlin ([4]).

Now put $\psi_i = B\varphi_i$, where $B$ is a linear operator acting from $X$ into $Y$. Equations (3) in this case take the form of equations of the N. M. Krylov method. Theorems 1 and 2 establish conditions for convergence of the method. Condition (2) in this case is fulfilled if the operators $A$ and $B$ satisfy the condition

$$(Au, Bu) \geqslant \gamma^2 \|y\|^2, \qquad \gamma = \text{const}. \tag{7}$$

Condition (7) is not difficult to verify in applications. We call this condition the $B$-positive definiteness of the operator $A$. If the operator $B$ has a bounded inverse, then instead of condition (7) one may verify the condition

$$(Au, Bu) \geqslant \alpha^2 \|Bu\|^2, \qquad \alpha = \text{const},$$

from which condition (7) follows.

If the operator $A$ is self-adjoint and positive-definite, then as the operator $B$ one may take, for example, even positive roots of the operator $A$ and, in particular, $B = A$. In comparison with the ordinary Galerkin method, this method may give the same advantage as the method

least squares in comparison with the Ritz method. Namely, in a number of cases one can directly assert a better character of convergence of the approximate solutions.

Let us consider, for example, the following boundary-value problem

$$Lu \equiv \Delta^2 u + \lambda K u = v, \qquad u\big|_S = 0, \qquad \frac{\partial u}{\partial n}\bigg|_S = 0, \tag{8}$$

where $Ku$ is an integro-differential expression of third order; $S$ is a sufficiently smooth boundary of a two- or three-dimensional domain $\Omega$. We consider equation (8) in $\mathring{W}_2^{(4)}$. From the embedding theorems [5] it follows that the operator $K$, defined by the expression $Ku$, as an operator acting from $\mathring{W}_2^{(4)}$ into $L_2$, will be completely continuous. The operator $\Delta^2$ has an inverse. Assuming that $\lambda$ is not an eigenvalue of the problem, and applying the method for finding the solution, we obtain a convergent process; moreover, putting $B = \Delta^2$ and denoting by $u$ the solution of the boundary-value problem, we shall have that $u_n \to u$ uniformly together with derivatives up to the second order, and in mean together with derivatives up to the fourth order—this follows from the embedding theorems. If it is assumed that the expression $Ku$ has order not higher than the second, then the usual Galerkin method may be applied. In this case we obtain convergence in mean up to the second order and uniform convergence of the functions themselves. The question of better convergence remains open.

Let us now note that if the operator $A$ is represented in the form of an odd power of some operator $T$, $A = T^{2k+1}$, and the operator $T^{2(k-r)}$ $(r < k)$ is positive-definite and self-adjoint, while the operator $T^{-1}$ exists and is bounded, then one may put $B = T^{2r+1}$ $(0 \leq r < k)$. This remark can be used in solving one-dimensional boundary-value problems of odd order.

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*Note: Figure translations are in progress. See original paper for figures.*