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Abstract

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ON EXPANSION IN EIGENFUNCTIONS OF SELF-ADJOINT OPERATORS

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1.

In a number of works ($^{1-4}$) the question of finding a complete system of generalized eigenfunctions of self-adjoint operators was considered. In essence, the main result of these works reduces to the following. In a Hilbert space H one chooses a certain subset—the basic functions. On it one introduces a new topology and defines functionals S that are continuous with respect to it; (φ, S) is the value of the functional S on the basic function φ. The functionals thus obtained are defined on the basic functions and are called **generalized functions**. Let A be a self-adjoint operator. As is known, Parseval' s formula holds:

$$(f, g) = \int \sum_n f_n(\lambda) \overline{g_n(\lambda)} \, d\sigma_\lambda$$

for any $f, g \in H$. Here σ_λ is the spectral function of the operator A; $f_n(\lambda)$ and $g_n(\lambda)$ are the Fourier transforms (constructed with respect to the operator A) of the elements f and g. It turns out that, whatever the self-adjoint operator A may be, one can, for almost all λ (with respect to the measure σ), find generalized functions $S_{n\lambda}$ such that for every basic function φ, for almost all λ, the equality

$$(\varphi, S_{n\lambda}) = \varphi_n(\lambda) \tag{1}$$

holds.

The generalized functions $S_{n\lambda}$ form a complete system of generalized eigenfunctions of the operator A.

I. M. Gel' fand and A. G. Kostyuchenko (^{4}) chose linear topological spaces as the space of basic functions. Later Yu. M. Berezanskii (^{3}) considered, as the Hilbert space, the space of square-summable functions of n variables, and as basic functions—finite functions differentiable n times, more precisely, finite functions admitting the application of the operator D

$$Df = \frac{\partial^n f}{\partial x_1 \, \partial x_2 \cdots \partial x_n}. \tag{2}$$

The methods applied in ([3]) do not require consideration of linear topological spaces. Yu. M. Berezanskii also considered some other function spaces (different from L_2) and, as D, some differential operators different from (2).

In the present note, similarly to what was done in ([3]), the basic functions are defined as the domains D_T of a certain linear operator T (which essentially plays the role of the operator D in ([3])); however, the operator T is no longer necessarily a differential operator, and the Hilbert space H, generally speaking, is not a function space. A new norm is introduced on the set D_T.

In order that the new norm and the original norm of the Hilbert space should turn out to be connected in a satisfactory way (for example, so that

for properties 1)–5) (see below) to hold, the following restrictions are imposed on the operators T:

A. T is a closed operator with dense domain D_T.

B. There exists a bounded operator T^{-1}, defined on the whole space, and $T^{-1}(T\varphi) = \varphi$ for all $\varphi \in D_T$.

Otherwise the operators T are completely arbitrary. Generalized functions (below they are called functionals generated by generalized elements) and basic functions (below they are called basic elements) form complete normed spaces.

Naturally, with such a broad definition of generalized elements, in general one cannot assert the existence of a complete system of generalized eigen-elements for an arbitrary self-adjoint operator.

In Theorems 1 and 3 it is asserted that, in order that among the generalized elements generated by the operator T there exist a complete system of eigen-elements (in the sense indicated above) of an arbitrary self-adjoint operator A, it is necessary and sufficient that the following condition be satisfied:

B. The operator T^{-1} has finite H-norm.

Recall that the H-norm of an operator B is equal to $\sum \|B\psi_\nu\|^2$, where $\{\psi_\nu\}$ is a complete orthonormal system of elements of the space H.

2. Below a construction of generalized elements is carried out which generalizes the construction of generalized functions of S. L. Sobolev.

Let T be an operator (in general, unbounded) in a separable Hilbert space H, satisfying conditions A, B. We shall call the domain D_T the **space of basic elements**. On D_T we define the functionals $(T\varphi, \hat{S})$, where $\hat{S}$ is an arbitrary fixed element of R_T (R_T is the range of the operator T). The functional so defined on D_T will be called the **functional generated by the generalized element** S, and the value of the functional on the basic element φ, ($\varphi \in D_T$), will be denoted by (φ, S). Then we have

$$(T\varphi, \hat{S}) = (\varphi, S) \tag{3}$$

The functional (φ, S), generally speaking, is not continuous (with respect to convergence in H) and cannot be extended to the whole space H. The generalized elements naturally form a linear space H_T—the **space of generalized elements**.

In the case when $\hat{S} \in D_{T^*}$, equality (3) may be rewritten in the form $(T\varphi, \hat{S}) = (\varphi, T^*\hat{S})$. In this case (and only in this case) the functional (φ, S) coincides (on D_T) with a continuous functional in H. We shall identify the generalized element S with the element $T^*\hat{S}$. It is not difficult to see that the whole Hilbert space H turns out to be a subset of H_T.

Introduce in D_T and H_T norms by putting $\|\varphi\|^* = \|T\varphi\|$ $(\varphi \in D_T)$; $\|S\|_* = \|\hat{S}\|$ $(S \in H_T)$.

Using conditions A and B, it is not difficult to prove that:

1) *With respect to the introduced norms, D_T and H_T form complete normed spaces.*
2) *Every functional $(\varphi, S) \in H_T$ is continuous with respect to the new norm introduced in D_T.*
3) *H_T is the conjugate space with respect to D_T.*
4) *The subset H is dense in H_T.*
5) *From the strong convergence of a sequence $\{\varphi_n\}$ to φ in the sense of convergence in D_T there follows the strong convergence of $\{\varphi_n\}$ to φ in the sense of convergence in H $(\varphi_n, \varphi \in D_T)$.*

Thus, for every operator T satisfying conditions A, B, there can be constructed a system of generalized elements, defined on D_T and possessing properties 1) ⋯ 5). This system is called the system generated by the operator T.

3. Theorem 1. Let T be an operator satisfying conditions A, B, and let H_T be the corresponding space of generalized elements; let A be an arbitrary self-adjoint operator with a resolution of the identity $E(\Delta)$ and spectral function $\sigma(\Delta)$. For every $f \in H$ there exists a set Λ_f of full σ-measure such that, for every $\lambda \in \Lambda_f$ and all $\varphi \in D_T$,

$$\lim_{n\to\infty} \frac{(\varphi, E(\Delta_\lambda^{(n)})f)}{\sigma(\Delta_\lambda^{(n)})} = (\varphi, f_\lambda), \tag{4}$$

where f_λ is some generalized element of H_T.

Here $\left(\Delta^{(n)}\right)$ is a regular sequence of nets (([5]), Ch. IV, § 15). In particular, one may take, for each n, $\Delta^{(n)}$ to be the system of half-intervals

$$\left[\frac{k}{n}, \frac{k+1}{n}\right)$$

(k integer, $-\infty < k < \infty$), covering the λ-axis; $\Delta_\lambda^{(n)}$ is the interval containing the point λ.

We note that from Plancherel' s formula and (4) the equality

$$(\varphi, f_\lambda) = \sum_n \varphi_n(\lambda)\overline{f_n(\lambda)}$$

easily follows for every $\varphi \in D_T$ for almost all λ (with respect to σ). If in this equality one takes as the element f the generating elements S_n, i.e. such that their Fourier transform $(S_n)_m(\lambda) = \delta_{nm}$, then one arrives at equality (1). This means that, in the case considered, every self-adjoint operator has a complete system of generalized eigen-elements.

Proof of Theorem 1. We shall show that the sequence of elements

$$((T^{-1})^* E(\Delta_\lambda^{(n)}) f)\, \sigma(\Delta_\lambda^{(n)})^{-1} \tag{5}$$

converges weakly (in the sense of convergence in H) for all $\lambda \in \Lambda_f$, where Λ_f is some set of full σ-measure. As is known,

$$(\psi, (T^{-1})^* E(\Delta) f) = (T^{-1}\psi, E(\Delta) f)$$

is an additive function of bounded variation. From a theorem on the differentiation of additive functions of bounded variation ([5], Ch. IV, No. 15) it follows that

$$\lim_{n\to\infty} (\psi, (T^{-1})^* E(\Delta_\lambda^{(n)}) f) \cdot \sigma(\Delta_\lambda^{(n)})^{-1} = L(\psi, f, \lambda)$$

exists and is finite almost everywhere for every $\psi \in H$. Consequently, there exists a set Λ'_f of full measure such that, for all $\lambda \in \Lambda'_f$ and for the sequence $\{\psi_\nu\}$, the limit $L(\psi_\nu, f, \lambda)$ exists and is finite. Here $\{\psi_\nu\}$ is an orthonormal basis in H.

To prove the assertion, it remains to verify that the norms of the sequence (5) are bounded uniformly in n. We estimate the squares of the norms of the sequence (5):

$$\left\|((T^{-1})^* E(\Delta_\lambda^{(n)}) f)\sigma(\Delta_\lambda^{(n)})^{-1}\right\|^2 = \sigma(\Delta_\lambda^{(n)})^{-2} \sum_\nu \left|(E(\Delta_\lambda^{(n)}) f,\, E(\Delta_\lambda^{(n)}) T^{-1}\psi_\nu)\right|^2 \leq$$

$$\leq \left(\sigma(\Delta_\lambda^{(n)})^{-1} \|E(\Delta_\lambda^{(n)}) f\|^2\right) \left(\sigma(\Delta_\lambda^{(n)})^{-1} \sum_\nu \|E(\Delta_\lambda^{(n)}) T^{-1}\psi_\nu\|^2\right) = A(n,\lambda) B(n,\lambda).$$

The functions

$$\|E(\Delta)f\|^2 \quad \text{and} \quad \sum_\nu \|E(\Delta)T^{-1}\psi_\nu\|^2$$

are additive nonnegative functions of bounded variation. The latter follows from the inequalities

$$\|E(\Delta)f\|^2 \leq \|f\|^2; \qquad \sum_\nu \|E(\Delta)T^{-1}\psi_\nu\|^2 \leq \sum_\nu \|T^{-1}\psi_\nu\|^2 < \infty$$

(by virtue of the finiteness of the H-norm of the operator T^{-1}). From the theorem on differentiation mentioned above ([5]) it follows that, for all λ belonging to some set Λ''_f of full σ-measure,

the limits $\lim_{n\to\infty} A(n,\lambda)$, $\lim_{n\to\infty} B(n,\lambda)$ exist and are finite, and therefore, for any $\lambda \in \Lambda'_f$, the norms of the elements (5) are bounded uniformly in n. This proves that on the set $\Lambda_f = \Lambda'_f \cap \Lambda''_f$ of full σ-measure the sequence (5) converges weakly. We denote its limit by F_λ.

By what has been proved, for all $\lambda \in \Lambda_f$ and $\psi \in H$ we have $L(\psi, f, \lambda) = (\psi, F_\lambda)$. Putting $\psi = T\varphi$, we obtain, for all $\varphi \in D_T$,

$$\lim_{n\to\infty} (\varphi, E(\Delta_\lambda^{(n)})f)\, \sigma(\Delta_\lambda^{(n)})^{-1} = \lim_{n\to\infty} (T\varphi, (T^{-1})^* E(\Delta_\lambda^{(n)})f)\, \sigma(\Delta_\lambda^{(n)})^{-1} =$$

$$= L(T\varphi, f, \lambda) = (T\varphi, F_\lambda).$$

Denoting the functional $(T\varphi, F_\lambda)$ by (φ, f_λ), we arrive at equality (4). The theorem is proved.

Under the assumption that the operator T still satisfies conditions A, B, the following theorem holds.

Theorem 2. *For any $\varphi, \varphi' \in D_T$ the equality*

$$(\varphi, \varphi') = \int (K_\lambda T\varphi, T\varphi')\, d\sigma_\lambda;$$

holds; K_λ, for almost all λ, is a positive operator with finite H-norm.

If H is a complete function space (for example L_2), then it follows from this that K_λ is an integral operator. Such a form of writing Plancherel' s formula for an arbitrary self-adjoint operator A was first obtained in ([3]) for the case $T = D$ (see ([2])).

4. Condition B is not only sufficient, but also necessary in order that, among the generalized elements of H_T, there should be contained a complete system of generalized eigenvectors of any self-adjoint operator. More precisely, the following theorem holds.

Theorem 3. *Let T be an arbitrary operator satisfying condition A and not satisfying condition B. Construct, from the operator T, the family H_T of generalized elements generated by it. There exists a self-adjoint operator A such that it is impossible to choose elements $S_{n\lambda} \in H_T$ so that, for every $\varphi \in D_T$, equality (1) should hold for almost every λ.*

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