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Abstract

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An Estimate of the Region of Convergence of Periodic Series–Solutions of Differential Equations with a Small Parameter

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1°. We consider, based on Lyapunov' s works (^{1,2}), a method for estimating the region of convergence of series that represent a periodic solution of a system of nonlinear differential equations obtained by Poincaré' s small-parameter method.

Let the equations for the deviations $x_1, \dots, x_n$ from the so-called "generating" periodic solution of the original system (nonautonomous) have the form

$$\frac{dx_s}{dt} = a_{s1}x_1 + \dots + a_{sn}x_n + \mu\varphi_s + X_s, \qquad s = 1, \dots, n, \tag{1}$$

where a_{sk} are constant coefficients; X_s are expanded in a certain domain into series (the analytic case) in integral positive powers of $x_1, \dots, x_n$ and of the parameter μ ($\mu > 0$), beginning with terms of the 2nd order, and φ_s and the coefficients of the expansions of X_s are continuous periodic functions of t with period 2π. Let the characteristic equation for the linear system obtained from (1) when $X_s = 0$ have r roots (simple*) of the form $\lambda_\sigma = \chi_\sigma\sqrt{-1}$ ($\sigma = 1, \dots, r$), where χ_σ is an integer or zero, while the remaining $n - r$ roots are not equal to zero or to an integral multiple of $\sqrt{-1}$. Such a case is called **resonant**, as distinct from the nonresonant case, when the indicated characteristic equation has no roots of the form $\chi_\sigma\sqrt{-1}$.

It may be assumed that the functions X_s have a factor μ (otherwise equation (1) can always be transformed to this form). Thus, the expansions of X_s may be written in the form

$$X_s = \mu\widetilde{X}_s,$$

where

$$\widetilde{X}_s = p_{s1}x_1 + \dots + p_{sn}x_n + \mu\psi_s + \sum_{m_1+\dots+m_{n+1}=2}^{\infty} P_s^{(m_1,\dots,m_{n+1})} x_1^{m_1} \cdots x_n^{m_n} \mu^{m_{n+1}}, \tag{2}$$

$P_s^{(m_1,\dots,m_{n+1})}$, p_{sk}, ψ_s being periodic coefficients.

The periodic solution of system (1), which tends to zero when $\mu = 0$, is sought in the form of series:

$$x_s = \mu x_s^{(1)} + \mu^2 x_s^{(2)} + \cdots, \qquad s = 1, \dots, n, \tag{3}$$

where $x_s^{(k)}$ are periodic functions with period 2π. The problem consists in estimating the domain of convergence of these series.

The equations for $x_s^{(1)}$ are written in the form

$$\frac{dx_s^{(1)}}{dt} = a_{s1} x_1^{(1)} + \cdots + a_{sn} x_n^{(1)} + \varphi_s, \qquad s = 1, \dots, n; \tag{4}$$

the equations for $x_s^{(k)}$, for any $k \geqslant 2$, are in the form of analogous linear nonhomogeneous equations

$$\frac{dx_s^{(k)}}{dt} = a_{s1} x_1^{(k)} + \cdots + a_{sn} x_n^{(k)} + f_s^{(k)},$$

$$f_s^{(k)} = p_{s1} x_1^{(k-1)} + \cdots + p_{sn} x_n^{(k-1)} + X_s^{(k-1)}, \tag{5}$$

where $X_s^{(k-1)}$ are finite combinations of those $x_1^{(\sigma)}, \dots, x_n^{(\sigma)}$ for which $\sigma \leqslant k - 2$.

If $\varphi_1, \dots, \varphi_n$ satisfy certain conditions, then a periodic solution of equations (4) exists and is found in the form

$$x_s^{(1)} = x_s^{(1)}(C) + \bar{x}_s^{(1)}, \tag{6}$$

where

$$x_s^{(1)}(C) = C_1^{(1)} K_{s1} e^{\lambda_1 t} + \cdots + C_r^{(1)} K_{sr} e^{\lambda_r t}; \qquad \bar{x}_s^{(1)} = J_s(\varphi_1, \dots, \varphi_n);$$

$K_{r\sigma}, C_\sigma^{(1)}$ are constants, with the $C_\sigma^{(1)}$ arbitrary; J_s are certain operators. These operators are such that $\bar{x}_s^{(1)}$ can be represented in the form of integrals of linear combinations of $\varphi_1, \dots, \varphi_n$ with variable coefficients. In finding $x_s^{(2)}$, the constants $C_\sigma^{(1)}$ do not, in general, remain arbitrary, but are determined from the

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* In general, it is immaterial whether these roots are simple or multiple. We regard them as simple only for t

conditions of periodicity of $x_s^{(2)}$. These conditions constitute algebraic linear nonhomogeneous equations with respect to $C_1^{(1)}, \dots, C_r^{(1)}$.

If the determinant of these equations is not equal to zero, then we obtain

$$C_\sigma^{(1)} = J_\sigma^c(\bar{f}_1^{(2)}, \dots, \bar{f}_n^{(2)}), \qquad \sigma = 1, \dots, r, \tag{7}$$

where J_σ^c are certain linear operators, and

$$\bar{f}_s^{(2)} = p_{s1}\bar{x}_1^{(1)} + \dots + p_{sn}\bar{x}_n^{(1)} + \psi_s.$$

Let us note that the nonvanishing of the indicated determinant and the conditions imposed on $\varphi_1, \dots, \varphi_n$ are sufficient for system (1), at any rate for sufficiently small μ, to have a periodic solution in the form of the series (3).

Suppose that all functions $x_1^{(\sigma)}, \dots, x_n^{(\sigma)}$, for $\sigma \leqslant k-1$, have turned out to be periodic, and that $x_s^{(k-1)}$ have the same form as $x_s^{(1)}$:

$$x_s^{(k-1)} = x_s^{(k-1)}(C) + \bar{x}_s^{(k-1)}, \qquad s = 1, \dots, n, \tag{8}$$

where

$$x_s^{(k-1)}(C) = C_1^{(k-1)} K_{s1} e^{\lambda_1 t} + \dots + C_r^{(k-1)} K_{sr} e^{\lambda_r t}, \qquad \bar{x}_s^{(k-1)} = J_s(f_1^{(k-1)}, \dots, f_n^{(k-1)}).$$

If the constants $C_\sigma^{(k-1)}$ are determined by formulas analogous to (7):

$$C_\sigma^{(k-1)} = J_\sigma^c(\bar{f}_1^{(k)}, \dots, \bar{f}_n^{(k)}), \qquad \sigma = 1, \dots, r, \tag{9}$$

where

$$\bar{f}_s^{(k)} = p_{s1}x_1^{(k-1)} + \dots + p_{sn}x_n^{(k-1)} + X_s^{(k-1)}, \qquad s = 1, \dots, n,$$

then the functions $x_s^{(k)}$ turn out to be periodic and are written in the same form as $x_s^{(k-1)}$.

With the aid of the expressions for the operators J_s, J_σ^c, one can estimate the functions $\bar{x}_s^{(k)}$ and the constants $C_\sigma^{(k-1)}$. We obtain

$$\max|\bar{x}_s^{(k)}| \le q_1^{(s)} \max|f_1^{(k)}| + \dots + q_n^{(s)} \max|f_n^{(k)}|, \qquad s = 1, \dots, n, \tag{10}$$

$$|C_\sigma^{(k-1)}| \le l_{\sigma 1} \max|\bar{f}_1^{(k)}| + \dots + l_{\sigma n} \max|\bar{f}_n^{(k)}|, \qquad \sigma = 1, \dots, r,$$

where $q_i^{(s)}$, $l_{\sigma i}$ are certain numbers.

Let us now form the algebraic equations

$$u_s = |K_{s1}|Y_1 + \dots + |K_{sr}|Y_r + \bar{u}_s, \qquad s = 1, \dots, n, \tag{11}$$

where

$$\bar{u}_s = \mu(\max|\bar{x}_s^{(1)}| + q_1^{(s)}U_1 + \dots + q_n^{(s)}U_n),$$

$$Y_\sigma = l_{\sigma 1}\bar{U}_1 + \dots + l_{\sigma n}\bar{U}_n, \qquad \sigma = 1, \dots, r. \tag{11*}$$

$U_s = U_s(u_1, \dots, u_n, \mu)$ are majorants of the expansions $\widetilde{X}_s$, and $\bar{U}_s$ are obtained from U_s after replacing $u_1, \dots, u_n$ by $\bar{u}_1, \dots, \bar{u}_n$ in the terms of first order with respect to $u_1, \dots, u_n$. These equations determine $u_1, \dots, u_n$ as implicit functions of the parameter μ.

If we seek a solution of equations (11) in the form of series

$$u_s = \mu u_s^{(1)} + \mu^2 u_s^{(2)} + \cdots, \qquad s = 1, \dots, n, \tag{12}$$

then, comparing the expressions for $u_s^{(k)}$ with the inequalities (10), one can verify that these series will majorize the series (3) under consideration. Thus, the problem has been reduced to determining the domain of existence of positive solutions of (11), vanishing for $\mu = 0$ and representable by series of the form (12).

Using the expressions (11*), we can transform (11) to the form

$$F_s = u_s - f_s(u_1, \dots, u_n, \mu), \qquad s = 1, \dots, n, \tag{13}$$

where the f_s expand into series with positive coefficients in integral positive powers of $u_1, \dots, u_n, \mu$; moreover, all f_s and $\partial f_s/\partial u_k$ vanish for $u_1 = \dots = u_n = \mu = 0$, but $f_s(0, \dots, 0, \mu) \neq 0$. It can be shown that (13) have the solution we need if $\mu \leq \bar{\mu}$, where $\bar{\mu}$ is found by solving the equations

$$F_s = 0, \qquad \frac{\partial(F_1, \dots, F_n)}{\partial(u_1, \dots, u_n)} = 0, \qquad s = 1, \dots, n, \tag{14}$$

which possess a unique system of positive roots $u_s = u_s^*$, $\mu = \bar{\mu}$. Thus, the series (4) converge in any case if $\mu \leq \bar{\mu}$.

2°. If the functions φ_s and $p_{sk}, \psi_s, P_s^{(m_1, \dots, m_{n+1})}$—the coefficients of the expansions X_s—are represented by Fourier polynomials, then the estimates (10) can be reached by finding the solution of equation (5) for different k by selecting it in the form of Fourier polynomials in complex form. Then we obtain all $\bar{x}_s^{(k)}$ and $C_\sigma^{(k-1)}$ in the form

$$\bar{x}_s^{(k)} = \sum_{j=-r_k}^{r_k} (q_{1j}^{(s)} A_{1j}^{(k)} + \dots + q_{nj}^{(s)} A_{nj}^{(k)}) e^{j\sqrt{-1}\,t}, \qquad s = 1, \dots, n, \tag{15}$$

$$C_\sigma^{(k-1)} = \sum_{m-1}^{r} (l_{\sigma 1}^{(m)} \bar{A}_{1xm}^{(k)} + \dots + l_{\sigma n}^{(m)} \bar{A}_{nxm}^{(k)}), \qquad \sigma = 1, \dots, r, \quad k = 2, 3, \dots .$$

where $q_{ij}^{(s)}, l_{\sigma i}^{(m)}$ are constants depending on the coefficients a_{sk} of the original system (1) and on the indices j, m, respectively, r_k depends on the index k, and

$$A_{sj}^{(k)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(p_{s1} x_1^{(k-1)} + \dots + p_{sn} x_n^{(k-1)} + X_s^{(k-1)}\right) e^{-ij\sqrt{-1}t}\, dt,$$

$$\bar{A}_{sxm}^{(k)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(p_{s1} \bar{x}_1^{(k-1)} + \dots + p_{sn} \bar{x}_n^{(k-1)} + X_s^{(k-1)}\right) e^{-\chi m\sqrt{-1}t}\, dt. \tag{15*}$$

On the basis of these expressions we obtain

$$\max |\bar{x}_s^{(k)}| \le q_1^{(s)} \sum_{j=-r_k}^{r_k} |A_{1j}^{(k)}| + \dots + q_n^{(s)} \sum_{j=-r_k}^{r_k} |A_{nj}^{(k)}|,$$

$$|C_\sigma^{(k-1)}| \le l_{\sigma 1} \sum_{m=1}^{r} |\bar{A}_{1xm}^{(k)}| + \dots + l_{\sigma n} \sum_{m=1}^{r} |\bar{A}_{nxm}^{(k)}|. \tag{16}$$

where $q_i^{(s)} \ge |q_{ij}^{(s)}|,\ l_{\sigma i} \ge |l_{\sigma i}^{(m)}|$ for all possible $j = 0,\ \pm 1, \dots, \pm r_k,\ m = 1, \dots, r$.

If it proves possible to establish that all $x_s^{(k)}$ contain no terms with $e^{ij\sqrt{-1}t}$, where $j = j_1,\ j = j_2, \dots$, then these values of j, in choosing the numbers $q_i^{(s)}, l_{\sigma s}$, are not taken into account. In this way the structure of the periodic solution being sought is, to a certain extent, taken into account.

After the coefficients $q_i^{(s)}, l_{\sigma i}$ of inequalities (16) have been found, one can form equations (11). The majorants U_s are obtained in this case by replacing, in the expansions of X_s, the functions $P^{(m_1,\dots,m_{n+1})}, p_{sk}, \psi_s$ by the sums of the absolute values of their (complex) Fourier coefficients.

3°. If the nonresonant case occurs, then all the preceding arguments remain valid; it is only necessary to put all $C_\sigma^{(k)}, Y_\sigma$ equal to zero and $x_s^{(k)} = \bar{x}_s^{(k)}, u_s = \bar{u}_s$.

Let us note that the method considered gives, in general, a better result than the method proposed by Lewis [3] and applicable only to the nonresonant case. For example, let the equation be given

$$\frac{d^2z}{dt^2} + \frac{1}{16}z = \mu(1 - z^2)\frac{dz}{dt} + \sin t, \tag{17}$$

which, for $\mu = 0$, has the "generating" periodic solution $z^0 = -\frac{16}{15}\sin t$. Putting $z = z^0 + x$, forming the equation for x, and investigating its periodic solution

$$x = \mu x^{(1)} + \mu^2 x^{(2)} + \cdots \tag{18}$$

as was indicated above, we obtain that this series converges, in any case, if $\mu < 0.15$. By the Lewis method, however, we obtain that convergence of the series (18) is guaranteed if approximately $\mu < 1/130$.

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CITED LITERATURE

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2. A. M. Lyapunov, "On series proposed by Hill for representing the motion of the Moon," Collected Works, 1, 1954, p. 418.
3. D. C. Lewis, Duke Math. J., 22, 1 (1955).

Note: Figure translations are in progress. See original paper for figures.

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