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Abstract

Full Text

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ON THE APPLICATION OF THE VARIATIONAL PRINCIPLE TO SOME NONLINEAR DIFFERENTIAL EQUATIONS

(Presented by Academician V. I. Smirnov on 14 III 1958)

1. Variational method.

It is required to solve the operator equation

$$Pu = f, \tag{1}$$

where P is a nonlinear operator in a Hilbert space H ; f is a prescribed element of this space. We shall suppose that the operator P is defined on some linear set M ; the solution of equation (1) will be sought in another linear set $M^0 \subset M$, with M^0 dense in H .

Theorem 1*. *Let:*

A. $P(0) = 0$; the Gâteaux differential $P'(x)y$ exists for arbitrary $x, y \in M$, is linear with respect to y , and, as an element of H , is continuous in every "plane" containing the point x .

B. $(P'(x)h_1, h_2) = (P'(x)h_2, h_1)$ for $x \in M$; $h_1, h_2 \in M^0$.

C. $(P'(x)h, h) > 0$ for $x \in M$; $h \in M^0$; $h \neq 0$.

If, moreover, there exists a solution of the equation

$$Pu = f, \quad u \in M^0,$$

then:

1) *this solution is unique;*

2) *it gives the functional*

$$\Phi(u) = \int_0^1 (Pt u, u) dt - (f, u) \tag{2}$$

a minimum value;

3) conversely, an element of the set M^0 realizing the minimum of the functional (2) satisfies equation (1).

The first assertion is verified directly. Indeed, assuming the existence of two distinct solutions $u_1, u_2 \in M^0$, we have the relation $(Pu_1 - Pu_2, u_1 - u_2) = 0$, which is transformed into the form

$$\int_0^1 (P'(tu_1 + (1-t)u_2)(u_1 - u_2), u_1 - u_2) dt = 0.$$

According to condition C, it follows from this that $u_1 - u_2 = 0$.

The proof of the second assertion is based on the formula

$$\Phi(u+h) - \Phi(u) = \int_0^1 (P(u+sh), h) ds - (f, h).$$

* In the examples considered by us, the operator P is differential; the set M is the domain of definition of the differential operations, while M^0 is determined by boundary conditions.

The third assertion has been proved repeatedly in theorems on the potentiality of operators ⁽¹⁾.

Theorem 2. Suppose that conditions A, B of Theorem 1 are satisfied, and also the condition

D. $(P'(x)h, h) \geq \gamma^2 \|h\|^2$ for all $x \in M$, $h \in M^0$.

Then the functional $\Phi(u)$ is bounded below. Every minimizing sequence converges in the metric of H .

Let us outline the proof of this theorem. With the aid of condition D it is easy to prove that $\text{Inf } \Phi(u) \geq -\frac{1}{2\gamma^2} \|f\|^2$. Obviously, the functional $\Phi(u)$ is convex; moreover, the inequality

$$\frac{1}{2}\Phi(u_k) + \frac{1}{2}\Phi(u_l) - \Phi\left(\frac{u_k + u_l}{2}\right) \geq \frac{\gamma^2}{8} \|u_k - u_l\|^2$$

holds for $u_k, u_l \in M^0$. It turns out that the left-hand side of this inequality tends to zero for any minimizing sequence $\{u_n\} \subset M^0$, if $k \rightarrow \infty$, $l \rightarrow \infty$. The limit of a minimizing sequence will be called a **generalized solution** of equation (1).

Theorem 3. *The generalized solution of problem (1) is unique. In other words: all minimizing sequences of the functional $\Phi(u)$ have one and the same limit in the space H .*

It may happen that the generalized solution found belongs to the set M^0 . Then it is the desired solution of equation (1). In the contrary case equation (1) has no solution on M^0 ; then the question arises of extending the domain of definition of the functional $\Phi(u)$, originally defined on the set M^0 .

2. Solution of a variational equation. Consider the functional $\Phi(u) = F(u) + lu$, defined on some linear set N of a Hilbert space H .

Theorem 4. *Suppose that the following conditions are satisfied:*

- 1) lu is a linear functional defined on H ;
- 2) $F(u) \geq \gamma^2 \|u\|^2$ for $u \in N$;
- 3) $F(2u) \leq kF(u)$, $k = \text{const}$, $u \in N$; $F(0) = 0$; $F(-u) = F(u)$;
- 4)

$$\begin{aligned} \rho(u, v) &= \frac{1}{2}\Phi(u) + \frac{1}{2}\Phi(v) - \Phi\left(\frac{u+v}{2}\right) = \frac{1}{2}F(u) + \frac{1}{2}F(v) - F\left(\frac{u+v}{2}\right) \geq \\ &\geq F\left(\frac{u-v}{2}\right), \quad u, v \in N; \end{aligned}$$

- 5) From $F(u-v) \rightarrow 0$ it follows that $|F(u) - F(v)| \rightarrow 0$, $u, v \in N$.

Then there exists a metric space $R \in H$, on which the functional $\Phi(u)$ attains its minimum. $\Phi(u)$ is continuous on R and N is dense in R .

Proof. $\rho(u, v) \geq 0$; putting here $u = x - y$, $v = z - x$, we obtain the estimate

$$F(x - y) + F(x - z) \geq 2F\left(\frac{y - z}{2}\right) \geq \frac{2}{k}F(y - z).$$

Here $k \geq 4$; this follows from the inequality $\rho(u, v) \geq F\left(\frac{u-v}{2}\right)$; for $v = 0$, $u = 2w$, $F(2w) \geq 4F(w)$. As a function of two points $x, y \in N$, the functional $F(x - y)$ satisfies the axioms of symmetry and identity of a metric space, as well as a weakened triangle condition. Chittenden proved ⁽⁵⁾ that under these conditions there exists on N a metric function $\delta(x, y)$, topologically equivalent to $F(x - y)$ and satisfying all the axioms of a metric space. Adjoining to N all limit elements in the sense of this metric, we obtain a complete metric space R . Consider a convergent sequence $\{u_k\} \subset R$. By assumption, $\delta(u_p, u_q) \rightarrow 0$ as $p \rightarrow \infty$, $q \rightarrow \infty$. Topological equivalence means that $F(u_p - u_q)$ also tends to zero. From condition 2) it follows,

that $\|u_p - u_q\| \rightarrow 0$, and therefore $R \subset H$. By condition 5), $|F(u_p) - F(u_q)| \rightarrow 0$; the sequence $\{F(u_n)\}$ has a finite limit, which we take as the value of $F(u)$ on

R . Thus the functional $\Phi(u)$ is extended to R uniquely. $\Phi(u_n)$ is bounded below for $u_n \in N$. Indeed,

$$\Phi(u_n) = F(u_n) + lu_n \geq \gamma^2 \|u_n\|^2 - \|l\| \|u_n\| \geq -\frac{1}{2\gamma^2} \|l\|^2;$$

as $n \rightarrow \infty$, $F(u_n) \rightarrow F(u)$, $lu_n \rightarrow lu$. Hence

$$\Phi(u) \geq -\frac{1}{2\gamma^2} \|l\|^2$$

for $u \in R$. Denote by d the lower bound of the functional $\Phi(u)$ on R . Then there exists at least one minimizing sequence $\{u_n\} \subset R$, $\lim_{n \rightarrow \infty} F(u_n) = d$. Suppose that, for $p, q > L$,

$$\Phi(u_p) < d + \varepsilon, \quad \Phi(u_q) < d + \varepsilon;$$

then $\rho(u_p, u_q) < \varepsilon$. Passing to the limit in condition 4), we note that $F(u_p - u_q) \rightarrow 0$ as $p \rightarrow \infty$, $q \rightarrow \infty$. The sequence $\{u_n\}$ converges in the metric R to some element $u \in R$. By definition,

$$F(u) = \lim_{n \rightarrow \infty} F(u_n) = d,$$

as was required to prove.

3. Examples.

1) The problem of elastic-plastic torsion of a solid rod.

This problem reduces to solving the equation

$$\frac{\partial}{\partial x} \left[f(T^2) \frac{\partial F}{\partial x} \right] + \frac{\partial}{\partial y} \left[f(T^2) \frac{\partial F}{\partial y} \right] = -2G\omega = \text{const}; \quad (3)$$

$$T^2 = (\text{grad } F)^2$$

with the boundary condition

$$u|_S = 0. \quad (4)$$

Problem (3)–(4) and the corresponding variational problem were indicated by L. M. Kachanov ⁽³⁾. The existence of a solution of problem (3)–(4) was proved by A. I. Koshelev for a smooth contour ⁽²⁾. It was assumed here that

$$f(T^2) = f_0 + \varphi(T^2), \quad f_0 > 0; \quad \varphi(t) \geq bt^\nu, \quad b > 0, \quad \nu > 0, \quad t > 0.$$

When the problem is solved by the variational method, the smoothness conditions on the contour are superfluous; as regards the behavior of the function $f(T^2)$, in addition to the corresponding smoothness conditions it is enough to impose the conditions

$$f(T^2) \geq f_0 > 0; \quad f(T^2) + 2f'(T^2)T^2 \geq \kappa > 0.$$

The proof of these assertions reduces to verifying conditions A, B, D.

2) The problem of creep of a plate clamped along the boundary

reduces to finding deflections $w(x, y)$ satisfying the equation

$$\begin{aligned} \frac{\partial^2}{\partial x^2} \left[g(H^2) \left(\frac{\partial^2 w}{\partial x^2} + \frac{1}{2} \frac{\partial^2 w}{\partial y^2} \right) \right] + \frac{\partial^2}{\partial y^2} \left[g(H^2) \left(\frac{\partial^2 w}{\partial y^2} + \frac{1}{2} \frac{\partial^2 w}{\partial x^2} \right) \right] + \\ + \frac{\partial^2}{\partial x \partial y} \left[g(H^2) \frac{\partial^2 w}{\partial x \partial y} \right] = p(x, y); \end{aligned} \quad (5)$$

$$H^2 = \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + \left(\frac{\partial^2 w}{\partial y^2} \right)^2 + \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2$$

under the boundary conditions

$$w|_S = 0, \quad \frac{\partial w}{\partial n} \Big|_S = 0. \quad (6)$$

Problem (5)–(6) was indicated by L. M. Kachanov⁽⁴⁾. Applying Theorems 1–3 to the problem formulated, we obtain the following conditions for its solvability by the variational method: a) $p(x, y) \in L_2(\Omega)$; b) the function $g(H^2)$ is three times continuously differentiable and satisfies the conditions

$$g(H^2) \geq g_0 > 0; \quad g(H^2) + 2g'(H^2)H^2 \geq \kappa > 0.$$

Analogous solvability conditions have been obtained in the plane problem of the theory of elastic-plastic deformations, as well as in the three-dimensional problem of elastic-plastic equilibrium, if the boundary conditions are given in displacements. Nonhomogeneous

the boundary conditions are then reduced to homogeneous ones with the aid of solutions of the corresponding linear problems of the theory of elasticity.

- 3) The variational equations of the considered problems of torsion of a bar and creep of a plate have the form

$$\delta\Phi(u) \equiv \delta \int_{\Omega} \left[\int_0^{J^2} \varphi(t) dt - 2q(x, y)u \right] d\Omega = 0.$$

In the torsion problem $\varphi(t) = f(t)$; $J^2 = (\text{grad } u)^2$; $q(x, y) = 2G\omega$, while in the problem of creep of a plate $\varphi(t) = g(t)$;

$$J^2 = \left(\frac{\partial^2 u}{\partial x^2} \right)^2 + \left(\frac{\partial^2 u}{\partial y^2} \right)^2 + \frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial^2 u}{\partial x \partial y} \right)^2; \quad q(x, y) = p(x, y).$$

Sometimes in the expansion of the function $\varphi(t)$ one restricts oneself to the linear approximation: $\varphi(t) = 1 + ct$, $c > 0$. A direct calculation shows that, for the linear approximation, the conditions of Theorem 4 are satisfied. As an equivalent metric one may choose the metric of the space $W_4^2(\Omega)$ in the torsion problem and the metric of the space $W_4^4(\Omega)$ in the problem of creep of a plate. Hence it follows, in particular, that the convergence of the variational method in the problems considered is better than in the corresponding problems of the linear theory of elasticity.

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