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## Abstract

## Full Text

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# A Well-Posed Boundary-Value Problem for Some Nonclassical Operators

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Hörmander ([1], see also [2]), using certain constructions proposed by M. I. Vishik ([3]), showed that in a bounded domain $V$, for any differential operator $a$ with constant coefficients, initially defined on the set of smooth functions identically equal to zero on the boundary of $V$, there exists a so-called solvable extension (i.e., an extension for which the equation $au = f$ has a unique solution for every right-hand side $f$ from the Hilbert space of square-summable functions in $V$). At the same time, however, the question of finding those conditions (boundary conditions) that determine this extension remains open. At present only the theory of a comparatively small number of boundary-value problems for equations of special types, generalizing the classical equations of mathematical physics, is sufficiently developed, and it is hardly possible to raise the question of a general method for finding a "good" problem for an arbitrary operator given in advance. In view of the above, the author finds it of interest to accumulate facts and methods relating to the selection of well-posed boundary-value problems for "nonclassical" operators. One such problem is considered below. The methods used are connected with ideas proposed in [4–6].

Let us consider a bounded star-shaped domain $Q$ of the $\nu$-dimensional space of variables $x_1, \dots, x_\nu$, and let $V = [0 \le x_0 \le 1] \times Q$. We shall study in $V$ the equation

$$au \equiv -D_0^3 u + bu = f, \qquad D_0 \equiv \frac{\partial}{\partial x_0}, \tag{1}$$

where $b$ is an elliptic (in the generalized sense) operator with constant coefficients, i.e., a differential operator of the form

$$b \equiv \sum_{|\alpha| \le m} b_\alpha D^{2\alpha}, \qquad D_\rho = \frac{\partial}{\partial x_\rho}, \qquad D^\alpha = D_1^{\alpha_1} \cdots D_\nu^{\alpha_\nu}, \qquad |\alpha| = \alpha_1 + \cdots + \alpha_\nu,$$

possessing definite definiteness properties formulated below. All derivatives are understood in the Sobolev–Schwartz sense. Neither the assumption that the coefficients are constant, nor the assumption of the absence of a non-self-adjoint lower part, is essential for the method used.

We shall regard as defined in $V$ the Hilbert space of square-summable functions with the usual scalar product and norm:

$$(u, v) = \int_V uv\, dV, \qquad \|u, H\|^2 = (u, u).$$

We shall say that $u \in \dot{C}$ if $u$ is a sufficiently smooth function identically equal to zero on the boundary of $V$. We assume the operator $b$ to be such that, for for $u \in \dot{C}$ the expression

$$(D_0 u, D_0 u) + (bu, u) = (D_0 u, D_0 u) + \sum_{|\alpha| \le m} (-1)^{|\alpha|} b_\alpha (D^\alpha u, D^\alpha u) \qquad (2)$$

may be regarded as defining a scalar square in a certain metric. The closure of $\dot{C}$ in the metric generated by the corresponding scalar product gives the Hilbert space $W$. We shall write the norm in $W$ in the form

$$|u, W|^2 = (D_0 u, D_0 u) + (Bu, Bu),$$

where the last term denotes an abbreviated notation for the sum on the right-hand side of (2). $B$ is understood as an operator assigning to a function $u$ a certain collection of its derivatives $Bu$. The functions in $W$ satisfy, on the lateral surface of the "cylinder" $V$, a definite system of homogeneous "boundary" conditions depending on the form of the operator $b$ and the domain $Q$. The nature of these conditions will be determined by the corresponding imbedding theorems. Their explicit form is of no interest to us.

**Example.** From our point of view, for instance, the operator

$$b \equiv -D_1^2 + D_2^4, \qquad (Bu, Bu) = (D_1 u, D_1 u) + (D_2^2 u, D_2^2 u)$$

is elliptic. When considered in $Q = [0 \le x_1 \le 1] \times [0 \le x_2 \le 1]$, the corresponding "boundary" conditions will be $u|_{x_1=0} = u|_{x_1=1} = u|_{x_2=0} = u|_{x_1=1} = 0$, $D_2 u|_{x_2=0} = D_2 u|_{x_2=1} = 0$.

We note that the conditions in $x_0$ satisfied by the functions in $W$ (i.e. the conditions $u|_{x_0=0} = u|_{x_0=1} = 0$) explicitly enter our considerations. We shall also use the fact that from $D_0^k w \in H$ it follows that $D_0^{k-1} w$ can be considered on an arbitrary section $x_0 = \text{const}$.

For an arbitrary $\tilde{u} \in W$, satisfying the additional conditions

$$D_0^2\tilde{u} \in H, \qquad D_0\tilde{u}|_{x_0=0} = 0, \tag{S}$$

define the operator $au$ as a functional on $W$, putting

$$\langle au, v\rangle = (D_0^2 u, D_0 v) + (Bu, Bv), \qquad v \in W. \tag{3}$$

We now construct the closure of the operator $a$ defined in this way. Consider on $W$ the functional defined by an element $f \in H$ according to the rule $\langle f, v\rangle = (f, v)$, with norm

$$|f, W^{-1}| = \sup_{v\in W} \frac{|(f, v)|}{|v, W|}. \tag{4}$$

The completion of $H$ in the norm (4) gives the space $W^{-1}$, isometrically isomorphic, as is not difficult to verify, to the space of all linear functionals on $W$, and therefore Hilbert.

Let now $u \in W$ be such that for it there exists an element $f \in W^{-1}$ and a sequence $u_i$ of functions from $W$, additionally satisfying condition (S) and the condition $D_0 B u_i \in H$ (which for the example considered means $D_0 D_1 u, D_0 D_2^2 u \in H$), such that $|u - u_i, W| \to 0,\ |au_i - f, W^{-1}| \to 0$ as $i \to \infty$.

We shall then call $u$ an $S$-solution of equation (1). The corresponding extension of the operator $a$ will be called the $S$-extension.

**Lemma 1.** For $S$-solutions of equation (1) the inequality

$$|u, W| \leq c|f, W^{-1}| \tag{}$$

holds.

It is sufficient to establish the inequality ($\Phi$) for the functions $u_i$ of the approximating sequence. The proof is carried out by substituting in (3) $v_i = u_i$, and then

$$v_i = v_{t,i} = \begin{cases} u_i(t,x)(1-x_0), & t \leq x_0 \leq 1, \\ u_i(x_0,x) - x_0 u_i(t,x), & 0 \leq x_0 \leq t, \end{cases} \qquad x = (x_1, \dots, x_n)$$

with subsequent integration with respect to $t$.

**Corollary 1.** The $S$-solution of equation (1) is unique in $W$.

**Corollary 2.** The range of the $S$-extension of the operator is closed in $W^{-1}$.

The next task is to show that the range of the $S$-extension $a$ coincides with all of $W^{-1}$, i.e., equation (1) has an $S$-solution for every $f \in W^{-1}$. For this we shall need to consider the operator $a^*$ (formally adjoint to the operator $a$) and the corresponding equation

$$a^* v = D_0^3 v + bv = g. \tag{1*}$$

Replacing the conditions ($S$) by the conditions ($S^*$)

$$D_0^2 v \in H, \qquad D_0 v\big|_{x_0=1} = 0, \tag{S*}$$

we can define $a^*$ as a functional on $W$, putting

$$\langle u, a^* v\rangle = -(D_0 u, D_0^2 v) + (Bu, Bv), \tag{3*}$$

where $u \in W$; $v \in W$ and satisfies the additional equation ($S^*$). For equation (1) there can be defined an $S$-solution for which the inequality ($\Phi$) will be valid. In general, every assertion proved for the operator $a$ automatically entails the validity of the corresponding assertion for the operator $a^*$. Therefore it is enough to restrict ourselves to the study of the operator $a$.

Let us define one more extension of the operator $a$, distinct from the $S$-extension. A function $u \in W$ for which there exists an element $f \in W^{-1}$ such that, for every $v \in W$ satisfying additionally the conditions ($S^*$) and the condition $D_0 Bv \in H$, the equality

$$\langle u, a^* v\rangle = \langle f, v\rangle$$

holds, will be called a $V$-solution of equation (1), and the corresponding extension of the operator $a$ a $V$-extension.

**Lemma 2.** *An element of $W$ orthogonal to the range of the $S$-extension of the operator $a^*$ is a $V$-solution of the equation*

$$au = 0. \tag{5}$$

The assertion follows immediately from the definitions.

**Lemma 3.** *The $V$-solution of equation (5) is unique and is equal to zero.*

For the proof, in (3) one substitutes the function $v$ determined by the conditions $D_0^3 v = u$; $v|_{x_0=0} = v_{x_0=1} = D_0 v|_{x_0=1} = 0$.

As noted above, the validity of Lemmas 2 and 3 entails the validity of Lemmas $2^*$ and $3^*$.

**Lemma $2^*$.** *An element of $W$ orthogonal to the range of the $S$-extension of the operator $a$ is a $V$-solution of the equation*

$$a^*v = 0. \tag{5*}$$

**Lemma** $3^*$**.** *The* $V$*-solution of equation* $(4^*)$ *is unique and is equal to zero.*

Hence we obtain:

**Corollary 3.** The range of the $S$-extension of the operator $a$ fills the entire space $W^{-1}$.

Since every $S$-solution is simultaneously a $V$-solution, and the latter, by Lemma 3, is unique, we have:

**Corollary 4.** For any $f \in W^{-1}$, the $S$-solution and the $V$-solution coincide.

We shall call the $S$-solution the **generalized solution** of equation (1).

**Theorem.** *The generalized solution of equation* (1) *(or* $(1^*)$*) exists for every* $f \in W^{-1}$ $(g \in W^{-1})$*, is unique, and belongs to* $W$*.*

The continuous dependence of the constructed solution on the right-hand side follows directly from $(\Phi)$.

**Remark.** From the coincidence of the $V$-solution and the $S$-solution it follows that every function $u \in W$ which, in addition, satisfies condition $(S)$, belongs to the domain of definition of the $S$-extension of the operator $a$.

**Concluding remark.** In our opinion, an essential difference between the constructions presented here (as well as those used in [6]) and the constructions in [4,5,7] is the use of the closure of the operator in the space of generalized functions $W^{-1}$. Usually the closure was constructed in $H$, and only in the investigation of the orthogonal complement to the range of the constructed extension (or of "weak solutions" of the adjoint homogeneous equation) were other spaces invoked [5]. The scheme used here makes it possible to bring "weak" $(V)$ and "strong" $(S)$ solutions as close together as possible, thereby facilitating the derivation of existence and uniqueness theorems.

We leave aside the question of the properties of the solution constructed.

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## CITED LITERATURE

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*Note: Figure translations are in progress. See original paper for figures.*