HYDROMECHANICS

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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

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ON THE NONUNIQUENESS OF THE SOLUTION OF THE HYDROFOIL PROBLEM

(Presented by Academician M. A. Lavrent' ev, 20 XI 1957)

In the linear formulation, the plane problem of hydrofoil theory has been the subject of numerous investigations. In the nonlinear formulation, for Froude numbers greater than 1, it was considered by A. M. Ter-Krikorov (^{1}). The nonlinear formulation makes it possible to establish a number of facts that cannot be detected in a linear treatment. The most interesting results are provided by the nonlinear theory for the study of flows at Froude numbers close to 1. Under these conditions it turns out that the flow problem has a nonunique solution. The existence of two flow regimes was apparently first established experimentally by G. S. Sukhomel (^{2}). Theoretically this fact was noted by N. N. Moiseev (^{3}) in the study of flow over an uneven bottom.

1. The plane problem of hydrofoil theory in dimensionless variables reduces to the determination of an analytic function $w(z)$ (see Fig. 1, where the notation is introduced), satisfying the flow conditions on L_0 and S, the asymptotic condition $\lim\limits_{x\to-\infty} y = 1$, and the condition of constancy of pressure on L, which is a streamline:

Fig. 1

$$\left|\frac{dw}{dz}\right|^2 + 2\gamma y = \text{const},$$

where $\gamma = gh^3/Q^2$; h is the depth of the fluid at infinity to the left; Q is the discharge.

Let, in the ζ-plane, the strip T_z correspond to a strip of unit width. If the contour S_1 (the image of the contour S) is prescribed, then the function $w(\zeta)$ can be determined. Denote

$$\left(\left|\frac{dw}{d\zeta}\right|^2\right)_{\eta=1} = 1+\omega.$$

The function $\zeta(z)$ will, for $\eta = 1$, satisfy the condition

$$\frac{1}{2(1+\omega)}\left|\frac{d\zeta}{dz}\right|^2 + \gamma y = \text{const}\,. \tag{1}$$

It can be shown that if $|1-\gamma|$ and $|\omega|$ are sufficiently small, then the bounda— the curve L satisfies the smoothness conditions under which M. A. Lavrent' ev' s estimates for $\left|\frac{d\zeta}{dz}\right|$ are valid*

$$\left|\frac{d\zeta}{dz}\right|^2 \sim \frac{1}{y^2}\left[1+\frac{2}{3}yy''\right].$$

Using this estimate, setting $y = 1+u$, and choosing the constant so that, for $\omega \equiv 0$, $u = 0$ satisfies equation (1), we obtain the following equation, which approximately describes the form of the free boundary:

$$u'' + 3(\nu-1)u + \frac{3}{2}(1+2\nu)u^2 = \mu\Phi, \tag{2}$$

where

$$\mu\Phi = \omega(x)[1+2\nu u^2 - (2\nu-1)u] + [\omega(\xi)-\omega(x)][1+2\nu u^2-(2\nu-1)u].$$

2. Let $\nu < 1$. To find a solution of equation (2) that would pass into the trivial one as $\mu \to 0$, put

$$u = \mu u_1 + \mu^2 u_2 + \dots, \qquad \Phi = \Phi_1 + \mu\Phi_2 + \dots \tag{3}$$

For the function u_1 we obtain the equation

$$u_1'' = 3(1-\nu)u_1 + \Phi_1. \tag{4}$$

Since, as $x \to -\infty$, $u_1 = 0$, denoting $\lambda = \sqrt{3(1-\nu)}$, we find:

$$u_1 = -\frac{1}{\lambda}\left\{e^{\lambda x}\int_x^\infty \Phi_1(\tau)e^{-\lambda\tau}\,d\tau + e^{-\lambda x}\int_{-\infty}^x \Phi_1(\tau)e^{\lambda\tau}\,d\tau\right\}.$$

The computation of the subsequent approximations is analogous. In this case the free surface of the flow has the form shown in Fig. 1.

Now let ν be close to 1. Denote $1-\nu=\alpha$, and write equation (2) in the form

$$u''=3\alpha u-\frac{9}{2}u^2+\mu f,$$

where $\mu f=\mu\Phi+\alpha\eta^2$, and put $u=u_0+\mu u_1+\ldots$. The function u_0 satisfies the equation

$$u_0''=3\alpha u_0-\frac{9}{2}u_0^2. \tag{5}$$

The solution of (5) possessing the prescribed asymptotics has the form

$$u_0=\frac{\alpha}{\operatorname{ch}^2\left\{\dfrac{\sqrt{3\alpha}}{2}(x+x_0)\right\}}, \tag{6}$$

where x_0 may be arbitrary.

Solution (6) describes a solitary wave. The function u_1 satisfies the equation

$$u_1''=3\alpha u_1-9u_0u_1+f_1(x,x_0,\alpha), \tag{7}$$

where $\mu f_1=\mu\Phi_1+\alpha u_0^2$.

Particular solutions of equation (7) for $f_1\equiv 0$ are

$$y_1=\frac{\operatorname{th}\dfrac{\sqrt{3\alpha}}{2}(x+x_0)}{\operatorname{ch}^2\dfrac{\sqrt{3\alpha}}{2}(x+x_0)};\qquad y_2=y_1\int^x\frac{\operatorname{ch}^4\dfrac{\sqrt{3\alpha}}{2}(\tau+x_0)}{\operatorname{th}^2\dfrac{\sqrt{3\alpha}}{2}(\tau+x_0)}\,d\tau.$$

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* See more details in (4).

Consequently, the general solution of equation (7) will be

$$u_1=\frac{1}{\Delta}\left\{y_1(x)\left[\int_0^x f_1(\tau,x_0,\alpha)y_2(\tau)\,d\tau+C_1\right]\right.$$
$$\left.-y_2(x)\left[\int_0^x f_1(\tau,x_0,\alpha)y_1(\tau)\,d\tau+C_2\right]\right\},$$

where $\Delta=y_2y_1'-y_1y_2'$.

The requirement that u_1 be bounded as $x \to \pm\infty$ leads to the equation

$$\int_{-\infty}^{\infty} f_1(\tau, x_0, \alpha) y_1(\tau)\, d\tau = 0; \tag{8}$$

since $\alpha = 1 - \nu$ is a quantity determining the flow velocity and is prescribed, from equation (8) we find the value of x_0. The constant C_1 may be arbitrary. To determine it one must consider the equations of the second approximation. The determination of the following terms of the expansion is analogous.

Thus a second solution has been found; it determines a flow whose free boundary passes into a solitary wave as $\mu \to 0$.

3. In exactly the same way one investigates the case $\nu > 1$. We shall seek the solution that passes into the trivial one in the form of the series (3). The equation of the first approximation for u_1 will be

$$u_1'' + 3(\nu - 1)u = \Phi_1.$$

The solution of this equation having the required asymptotic behavior as $x \to -\infty$ has the form

$$u_1 = \frac{1}{\sqrt{3(\nu-1)}} \int_{-\infty}^{x} \overline{\Phi}(\tau) \sin \sqrt{3(\nu-1)}(x - \tau)\, d\tau. \tag{9}$$

The shape of the free surface is shown in Fig. 2. As $x \to +\infty$, the free surface tends to a regular sinusoid of the form $u_1 = A \sin \sqrt{3(\nu-1)}\, x + B \cos \sqrt{3(\nu-1)}\, x$. To construct a second solution, different from (9), we set, as before, $\nu - 1 = \alpha$, $u = u_0 + \mu u_1 + \cdots$. The function u_0 will satisfy the equation

$$u_0'' - 3\alpha u_0 - \frac{9}{2} u_0^2 = 0.$$

This equation has the solution

$$u_0 = \alpha \left[\frac{1}{\operatorname{ch}^2 \frac{\sqrt{3\alpha}}{2}(x - x_0)} - \frac{2}{3} \right], \tag{10}$$

which also describes a solitary wave, whose depth as $x \to \pm\infty$ will be $h^* = h\left[1 - \frac{2(\nu-1)}{3}\right]$, where h is the depth of the solitary wave found in the preceding section. Starting from the solution (10) and repeating the arguments given above, we shall find a flow which, as $\mu \to 0$, passes into the flow whose free boundary is described by equation (10).

Fig. 2

Fig. 2

Figure 2: Fig. 2

Fig. 3

Figure 3: Fig. 3

4. Thus, even the simplest analysis of the nonlinear problem reveals a number of qualitative features of the hydrofoil which in principle cannot be studied within the framework of the linear theory. The qualitative picture of the nonuniqueness can be represented schematically in the plane αQ (amplitude–discharge). In Fig. 3, curve I is the solution (3) for $\nu < 1$; II is a solution of the form $u = u_0 + \mu u_1$, where u_0 is determined by formula (10); III is a solution of the form $u = u_0 + \mu u_1$, where u_0 is determined by formula (6); IV is a solution of the form (9).

Fig. 3

5. Different flow regimes also correspond to different hydrodynamic characteristics of the hydrofoil. As an example, let us consider the motion of a vortex under the surface of a liquid. Let its coordinates in the ζ-plane be $(0, \frac{1}{2})$; then the function $w(\zeta)$ has the form

$$w(\zeta) = \zeta + \frac{\Gamma}{2\pi i} \ln \frac{\operatorname{sh} \frac{\pi}{2}\left(\zeta - \frac{i}{2}\right)}{\operatorname{sh} \frac{\pi}{2}\left(\zeta + \frac{i}{2}\right)}.$$

Then

$$\omega(\xi) = \frac{\Gamma}{2 \operatorname{ch} \pi\xi}.$$

Using the formulas of S. A. Chaplygin, it is not difficult to determine the forces acting on the vortex. If $\nu < 1$, then for the lift force we obtain the following formulas (written in dimensional form):

$$Y = -c\rho\Gamma - \frac{c\rho\Gamma^2}{4h\sqrt{3(c^2 - gh)}}; \tag{I}$$

$$Y = -c\rho\Gamma + \frac{\rho\Gamma}{c}(c^2 - gh); \tag{II}$$

here c is the velocity of motion of the vortex.

The first formula determines the lift force in a flow "close" to uniform. The second formula gives the value of the lift force in a flow "close" to a flow whose free surface is a solitary wave.

Remark. Since the contour is specified in the parametric plane, it is clear that the solutions found correspond, generally speaking, to different foils. Nevertheless, the calculations presented indicate the possibility of the existence of two forms of flow.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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