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I. I. OGIEVETSKII

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Abstract

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MATHEMATICS

I. I. OGIEVETSKII

ON THE THEORY OF FRACTIONAL DIFFERENTIATION AND INTEGRATION OF PERIODIC FUNCTIONS BELONGING TO THE CLASS $L_p,\ p > 1$

(Presented by Academician S. N. Bernstein on 15 VII 1957)

The definition of the fractional integral (derivative) of a periodic function, given by Weyl in ([1]) (see also ([2])), was studied in detail by Hardy and Littlewood in the memoir ([3]), which contains, in particular, a number of results on the passage of functions belonging to Lipschitz classes into other classes of the same type as a consequence of their integration and differentiation of fractional order. Some of these results were supplemented by Zygmund in ([4]), owing to his introduction of the so-called functions of the Zygmund class (for their definition see ([4]); in the literature they are sometimes called "quasi-smooth").

The present paper is carried out on the basis of a method for investigating integrals and derivatives of fractional order. This method is based on the study of the change in the order of the best approximation of a periodic function belonging to the class $L_p,\ p > 1$, as a consequence of its integration and differentiation of fractional order. In the paper there is given, in a certain sense, an exhaustive consideration of the above-mentioned assertions on the passage of functions of the Lipschitz and Zygmund classes into other classes of the same kind as a consequence of their integration and differentiation of fractional order, the simplest examples of which were indicated in the above-cited works of Hardy, Littlewood, and Zygmund.

Let $E_n^p f(x)$ denote the best approximation of the function $f(x)$ in the metric L_p by a trigonometric polynomial $T_n(x)$

$$E_n^p f(x) = \min_{T_n} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x) - T_n(x)|^p \, dx \right\}^{1/p}.$$

Let $f(x)$ have period 2π and

$$\int_0^{2\pi} f(x)\,dx = 0.$$

We shall denote by $f_\alpha(x)$ the fractional integral of order α ([2,4])

$$f_\alpha(x) = \cos\frac{\pi\alpha}{2}\sum_{\nu=1}^{\infty}\frac{A_\nu(x)}{\nu^\alpha} + \sin\frac{\pi\alpha}{2}\sum_{\nu=1}^{\infty}\frac{B_\nu(x)}{\nu^\alpha},$$

where $\sum_{\nu=1}^{\infty} A_\nu(x)$ is the Fourier series of the function $f(x)$, and $\sum_{\nu=1}^{\infty} B_\nu(x)$ is the series conjugate to the Fourier series of the function $f(x)$. We shall denote by $f^\alpha(x) = f_{-\alpha}(x)$ the fractional derivative of order α of the function $f(x)$.

The following symmetric assertions are valid:

Theorem 1. From $E_n^p f(x) = O(1/n^\beta)$, $\beta > 0$, it follows that $E_n^p f_\alpha(x) = O(1/n^{\alpha+\beta})$.

Theorem 2. From $E_n^p f(x) = O(1/n^\beta)$, $0 < \gamma < \beta$, it follows that $f^\gamma(x)$ exists, and moreover $E_n^p f^\gamma(x) = O(1/n^{\beta-\gamma})$.

The second of these assertions may be regarded as a generalization of the well-known converse theorem of S. N. Bernstein for L_p-integrable periodic functions ([8]) to the case of an arbitrary fractional index of differentiation.

For the case of the C-metric, analogous results were established in ([6]). The proof of these results for the case of the L_p-metric considered in the present paper requires the use of other methods, partly related to some of the methods used in ([7,9]), and rests, in particular, on the following lemma of the type repeatedly applied in the work of Alexits ([9]).

Lemma. Let $T_\nu(x)$ be a sequence of trigonometric polynomials of order $\nu - 20$, and suppose, in addition, that in the space $L_p(-\pi, \pi)$

$$\left\|\sum_{\nu=n+1}^{m} T_\nu(x)\right\|_p = O(\lambda_n),$$

$$\alpha > 0, \quad 1 < p \le p' \le \infty, \quad \alpha - \frac{1}{p} + \frac{1}{p'} > 0,$$

where λ_n is a certain sequence of real numbers such that $\lambda_n = o(n^{\alpha - 1/p + 1/p'})$. Then the series

$$\sum_{k=1}^{\infty}\frac{T_k(x)}{k^\alpha}$$

converges in the metric $L_{p'}$ to $\Phi(x) \in L_{p'}$, and

$$\left\| \Phi(x) - \sum_{k=1}^{n} \frac{T_k(x)}{k^\alpha} \right\|_{p'} = O\left(\frac{\lambda_n}{n^{\alpha-1/p+1/p'}} \right).$$

In proving the lemma, one uses the well-known inequality of S. M. Nikol' skii, which relates the norms of a trigonometric polynomial in different spaces L_p ([10]).

Let us show how, using Theorems 1 and 2 and the known characteristics of the Lipschitz classes $\mathrm{Lip}(\alpha, p)$ and Zigmund' s Λ_p^* in terms of constructive function theory ($f(x) \in \mathrm{Lip}(\alpha, p)$ is equivalent to $E_n^p f(x) = O(1/n^\alpha)$ ([5]), and $f(x) \in \Lambda_p^*$ is equivalent to $E_n^p f(x) = O(1/n)$ ([4])), one can obtain the above-mentioned results of Zygmund (for $p > 1$), given by him without proof in ([4]) (p. 69) and formulated in the following way.

Let $0 < \alpha < 1$. Then from $f(x) \in \mathrm{Lip}(\alpha, p)$ it follows that $f_{1-\alpha}(x) \in \Lambda_p^*$. If $f(x) \in \Lambda_p^*$, then $f_{1-\alpha}(x)$ is continuous and $df_{1-\alpha}/dx = f^\alpha(x)$ exists almost everywhere and belongs to the class $\mathrm{Lip}(1-\alpha, p)$. Indeed, from $f(x) \in \mathrm{Lip}(\alpha, p)$ it follows that $E_n^p f(x) = O(1/n^\alpha)$ and hence (Theorem 1), $E_n^p f_{1-\alpha}(x) = O(1/n)$, i.e. $f_{1-\alpha}(x) \in \Lambda_p^*$. Let $f(x) \in \Lambda_p^*$; then $E_n^p f(x) = O(1/n)$, and hence $E_n^p f_{1-\alpha}(x) = O(1/n^{2-\alpha})$. From this we conclude (Theorem 2) that $df_{1-\alpha}/dx$ exists and

$$E_n^p \left(\frac{df_{1-\alpha}}{dx} \right) = O\left(\frac{1}{n^{1-\alpha}} \right),$$

i.e.

$$\frac{df_{1-\alpha}}{dx} \in \mathrm{Lip}(1-\alpha, p);$$

since $df_{1-\alpha}/dx \in L_p$, $p > 1$, the function $df_{1-\alpha}/dx$ is integrable and its integral coincides with $f_{1-\alpha}(x)$, and, consequently, $f_{1-\alpha}(x)$ is absolutely continuous.

In an analogous way one can show that the following theorem is valid:

Theorem 3. The following assertions are valid:

I. From $f^\beta \in \mathrm{Lip}(\alpha, p)$ it follows that:

$$(f_\gamma)^\delta \subset \mathrm{Lip}(\alpha+\beta+\gamma-\delta, p) \quad \text{if } 0 < \alpha+\beta+\gamma-\delta < 1; \tag{1}$$

$$(f_\gamma)^\delta \subset \Lambda_p^* \quad \text{if } \alpha+\beta+\gamma-\delta = 1; \tag{2}$$

$$(f^\gamma)_\delta \subset \mathrm{Lip}(\alpha+\beta+\delta-\gamma, p) \quad \text{if } 0 < \alpha+\beta+\delta-\gamma < 1,\ \gamma < \alpha+\beta; \tag{3}$$

$$(f^\gamma)_\delta \subset \Lambda_p^* \quad \text{if } \alpha+\beta+\delta-\gamma = 1,\ \gamma < \alpha+\beta; \tag{4}$$

II. From $f^\beta \in \Lambda_p^*$ it follows that:

$$(f_\gamma)^\delta \subset \mathrm{Lip}(1+\beta+\gamma-\delta, p) \quad \text{if } 0<1+\beta+\gamma-\delta<1; \tag{1}$$

$$(f_\gamma)^\delta \subset \Lambda_p^* \quad \text{if } \delta=\beta+\gamma; \tag{2}$$

$$(f^\gamma)_\delta \subset \mathrm{Lip}(1+\beta+\delta-\gamma, p) \quad \text{if } 0<1+\beta-\gamma<1,\ \gamma<1+\beta; \tag{3}$$

$$(f^\gamma)_\delta \subset \Lambda_p^* \quad \text{if } \gamma=\beta+\delta,\ \gamma<1+\beta. \tag{4}$$

III. From $f_\beta \in \mathrm{Lip}(\alpha, p)$ it follows that:

$$(f_\gamma)^\delta \subset \mathrm{Lip}(\alpha-\beta+\gamma-\delta, p) \quad \text{if } 0<\alpha-\beta+\gamma-\delta<1; \tag{1}$$

$$(f_\gamma)^\delta \subset \Lambda_p^* \quad \text{if } \alpha-\beta+\gamma-\delta=1,\ \beta<\alpha; \tag{2}$$

$$(f^\gamma)_\delta \subset \mathrm{Lip}(\alpha-\beta-\gamma+\delta, p) \quad \text{if } 0<\alpha-\beta-\gamma+\delta<1,\ \beta<\alpha; \tag{3}$$

$$(f^\gamma)_\delta \subset \Lambda_p^* \quad \text{if } \alpha-\beta-\gamma+\delta=1,\ \beta<\alpha; \tag{4}$$

IV. From $f_\beta \in \Lambda_p^*$ it follows that:

$$(f_\gamma)^\delta \subset \mathrm{Lip}(1-\beta+\gamma-\delta, p) \quad \text{if } 0<1-\beta-\gamma+\delta<1,\ \beta<1; \tag{1}$$

$$(f_\gamma)^\delta \subset \Lambda_p^* \quad \text{if } \gamma=\beta+\delta,\ \beta<1; \tag{2}$$

$$(f^\gamma)_\delta \subset \mathrm{Lip}(1-\beta-\gamma-\delta, p) \quad \text{if } 0<1-\beta-\gamma+\delta<1,\ \beta<1; \tag{3}$$

$$(f^\gamma)_\delta \subset \Lambda_p^* \quad \text{if } \delta=\beta+\gamma,\ \beta<1. \tag{4}$$

For the case of the C-metric this theorem was established in ([6]).

It is not difficult to see that the results of Zygmund proved above are contained in Theorem 3 for a suitable choice of the parameters. Indeed, to obtain the first of Zygmund' s assertions one should put in (I, 2) $\beta=0,\ \delta=0,\ \gamma=1-\alpha$, and for the second–put in (II, 1) $\beta=0,\ \delta=1,\ \gamma=1-\alpha,\ 0<\alpha<1$.

Putting $\beta = 0,\ \delta = 0$ in (I, 1) and in (I, 3), we obtain, respectively, that from $f(x) \in \mathrm{Lip}(\alpha, p)$ it follows that $f_\gamma(x) \in \mathrm{Lip}(\alpha + \gamma, p)$, $0 < \alpha + \gamma < 1$, and that from $f(x) \in \mathrm{Lip}(\alpha, p)$ it follows that $f^\gamma(x) \in \mathrm{Lip}(\alpha - \gamma, p)$, $0 < \gamma < \alpha < 1$, i.e. the well-known results of Hardy and Littlewood ([3,4]). Let us note that a direct derivation of these assertions of Hardy and Littlewood, also based on methods of constructive function theory, is given in ([7]).

Using the lemma, by a method analogous to that by which Theorem 1 was proved, one can establish Theorem 4.

Theorem 4. From

$$E_n^p f(x) = O(1/n^\beta), \quad \beta > 0,\ \alpha > 0,\ \alpha - 1/p + 1/p' > 0, \quad 1 < p \le p' \le \infty,$$

it follows that

$$\left\| f_\alpha(x) - s_n(f_\alpha, x) \right\|_{p'} = O\!\left(\frac{1}{n^{\alpha+\beta-1/p+1/p'}}\right).$$

Using Theorems 2 and 4, we arrive at Theorem 5.

Theorem 5. If

$$E_n^p f(x) = O(1/n^\beta), \quad 0 < \alpha < \beta,\ \beta - \alpha - 1/p + 1/p' > 0, \quad 1 < p \le p' \le \infty,$$

then

$$\left\| f^\alpha(x) - s_n(f^\alpha, x) \right\|_{p'} = O\!\left(\frac{1}{n^{\beta-\alpha-1/p+1/p'}}\right).$$

Using the lemma, one can show the validity of the following theorem.

Theorem 6. Let $f(x) \in L_p,\ p > 1,\ \alpha > 0,\ \alpha - 1/p + 1/p' > 0,\ 1 < p \le p' \le \infty$. Then

$$\| f_\alpha(x) - s_n(f_\alpha, x) \|_{p'} = O\left(\frac{1}{n^{\alpha-1/p+1/p'}}\right).$$

Remark. Theorems 4, 5, 6 in the limiting case $p' = \infty$ and under the assumptions: $0 < \beta < 1,\ \alpha + \beta < 1$ for Theorem 4; $0 < \beta < 1$ for Theorem 5; and $1/p < \alpha < 1$ for Theorem 6 were established in the work of Izumi and Sato [(11)].

From Theorems 4, 5, 6, using the characteristics of the Lipschitz and Zygmund classes in terms of the constructive theory of functions, there follow several interesting corollaries on the passage of functions belonging to the Lipschitz classes $\mathrm{Lip}(\alpha, p)$ and the Zygmund classes Λ_p^* into other classes of the same kind (Theorems 7, 8, and 9).

Theorem 7. Let $f(x) \in L_p,\ 1 < p \le p' \le \infty$. Then:

$$f_\alpha(x) \in \mathrm{Lip}\left(\alpha - \frac{1}{p} + \frac{1}{p'}, p'\right), \quad \text{if } 0 < \alpha - \frac{1}{p} + \frac{1}{p'} < 1;$$

$$f_\alpha(x) \in \Lambda^*_{p'}, \quad \text{if } \alpha = 1 + \frac{1}{p} - \frac{1}{p'}.$$

For $p' = \infty$ and $\alpha - 1/p < 1$, the first result was established by Hardy and Littlewood [2,3].

Theorem 8. Let $f(x) \in \mathrm{Lip}(\alpha, p),\ 0 < \alpha \leq 1$. Then:

$$f_\beta(x) \in \mathrm{Lip}\left(\alpha + \beta - \frac{1}{p} + \frac{1}{p'}, p'\right), \quad \text{if } 0 < \alpha + \beta - \frac{1}{p} + \frac{1}{p'} < 1;$$

$$f_\beta(x) \in \Lambda^*_{p'}, \quad \text{if } \alpha + \beta - \frac{1}{p} - \frac{p}{p'} = 1.$$

If $f(x) \in \Lambda^*_p$, then

$$f_\beta(x) \in \mathrm{Lip}\left(1 + \beta - \frac{1}{p} - \frac{1}{p'}, p'\right), \quad \text{if } 0 < 1 + \beta - \frac{1}{p} + \frac{1}{p'} < 1;$$

$$f_\beta(x) \in \Lambda^*_{p'}, \quad \text{if } \beta = \frac{1}{p} + \frac{1}{p'}.$$

Theorem 9. If $f(x) \in \mathrm{Lip}(\alpha, p),\ 0 < \alpha \leq 1$, then

$$f^\gamma(x) \in \mathrm{Lip}\left(\alpha - \gamma - \frac{1}{p} + \frac{1}{p'}, p'\right), \quad \text{if } \alpha - \gamma - \frac{1}{p} + \frac{1}{p'} = 0.$$

If $f(x) \in \Lambda^*_p$, then

$$f^\gamma(x) \in \mathrm{Lip}\left(1 - \gamma - \frac{1}{p} + \frac{1}{p'}, p'\right), \quad \text{if } 1 - \gamma - \frac{1}{p} + \frac{1}{p'} > 0.$$

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References Cited

[1] H. Weyl, *Vierteljahresschrift d. Naturforsch. Ges. Zürich*, **62**, 296 (1917).
[2] A. Zygmund, *Trigonometric Series*, 1939.
[3] G. H. Hardy, J. E. Littlewood, *Math. Zs.*, **27**, 565 (1928).
[4] A. Zygmund, *Duke Math. J.*, **12**, 47 (1945).
[5] E. S. Quade, *Duke Math. J.*, **3**, 529 (1937).
[6] I. I. Ogievetskii, *Ukr. Mat. Zh.*, **9**, no. 3 (1957).
[7] D. Králik, *Acta Math. Acad. Sci. Hung.*, **7**, 49 (1956).
[8] N. I. Akhiezer, *Lectures on Approximation Theory*, 1947.

[9] G. Alexits, *Acta Math. Acad. Sci. Hung.*, **3**, 29 (1952).
[10] S. M. Nikol' skii, *Tr. Mat. Inst. im. V. A. Steklova*, **37**, 244 (1951).
[11] S. Isumi, M. Sato, *Proc. Japan. Acad.*, **31**, 659 (1955).

Note: Figure translations are in progress. See original paper for figures.