UNSTEADY FLOW WITH HEAT TRANSFER IN A VISCOUS INCOMPRESSIBLE FLUID BETWEEN TWO ROTATING DISKS IN THE PRESENCE OF INJECTION

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Abstract

Full Text

FLUID MECHANICS

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UNSTEADY FLOW WITH HEAT TRANSFER IN A VISCOUS INCOMPRESSIBLE FLUID BETWEEN TWO ROTATING DISKS IN THE PRESENCE OF INJECTION

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1. Let us consider the problem of an unsteady flow of a viscous incompressible fluid, arising from a state of rest, between two infinite disks separated from one another by a distance h, of which one rotates with a time-dependent angular velocity $\omega_0(t)$, and the other with angular velocity $\omega_1(t)$. Let uniform injection of the same fluid take place from the first disk with a time-dependent velocity $v_0(t)$, and from the second with velocity $v_1(t)$.

If v_r, v_θ, v_z are, respectively, the radial, tangential, and axial components of the velocity vector, then the boundary conditions of the problem, determined by the no-slip condition and by the presence of injection, and the initial conditions, determined by the absence of initial velocity of motion, are written in the form

$$v_r(r,0,t) = v_r(r,h,t) = 0, \tag{1}$$

$$v_\theta(r,0,t) = r\omega_0(t), \qquad v_\theta(r,h,t) = r\omega_1(t); \tag{2}$$

$$v_z(r,0,t) = v_0(t), \qquad v_z(r,h,t) = v_1(t); \tag{3}$$

$$v_r(r,z,0) = v_\theta(r,z,0) = v_z(r,z,0) = 0. \tag{4}$$

It is easy to verify that the Navier–Stokes equations, written in a cylindrical coordinate system, in the presence of axial symmetry and absence of body forces, and under conditions (1)–(4), admit the following solution:

$$v_r = \frac{r}{t_0}F(\zeta,\tau), \qquad v_\theta = \frac{r}{t_0}G(\zeta,\tau), \qquad v_z = \sqrt{\frac{\nu}{t_0}}\,H(\zeta,\tau),$$

$$P\frac{t_0}{\rho\nu} = \frac{1}{2}A(\tau)\frac{r^2}{\nu t_0} + B(\zeta,\tau), \qquad \zeta = \frac{z}{\sqrt{\nu t_0}}, \qquad \tau = \frac{t}{t_0}, \tag{5}$$

where t_0 is a certain constant with the dimension of time, ν is the coefficient of kinematic viscosity, P is the pressure, and ρ is the density.

In this case the functions F, G, and H must satisfy the system of nonlinear partial differential equations

$$\frac{\partial^4 H}{\partial\zeta^4} = H\frac{\partial^3 H}{\partial\zeta^3} + 4G\frac{\partial G}{\partial\zeta} + \frac{\partial^3 H}{\partial\zeta^2\partial\tau}, \tag{6}$$

$$\frac{\partial^2 G}{\partial\zeta^2} = H\frac{\partial G}{\partial\zeta} - \frac{\partial H}{\partial\zeta}G + \frac{\partial G}{\partial\tau}, \qquad 2F + \frac{\partial H}{\partial\zeta} = 0. \tag{7}$$

with the following boundary and initial conditions:

$$H(0,\tau) = \sqrt{\frac{t_0}{\nu}}\,v_0(t) = V_0(\tau), \qquad H(\zeta_1,\tau) = \sqrt{\frac{t_0}{\nu}}\,v_1(t) = V_1(\tau); \tag{8}$$

$$\frac{\partial H}{\partial\zeta}(0,\tau) = \frac{\partial H(\zeta_1,\tau)}{\partial\zeta} = 0. \tag{9}$$

$$G(0,\tau) = \omega_0(t)t_0 = \Omega_0(\tau), \qquad G(\zeta_1,\tau) = \omega_1(t)t_0 = \Omega_1(\tau), \tag{10}$$

$$G(\zeta,0) = H(\zeta,0) = 0, \qquad \zeta_1 = \frac{h}{\sqrt{\nu t_0}}. \tag{11}$$

The functions $A(\tau)$ and $B(\zeta,\tau)$ are determined, after solving the problem (6)–(11), from the equations

$$A(\tau) = \frac{\partial^2 F}{\partial\zeta^2} + G^2 - F^2 - H\frac{\partial F}{\partial\zeta} - \frac{\partial F}{\partial\tau}, \qquad \frac{\partial B(\zeta,\tau)}{\partial\zeta} = \frac{\partial^2 H}{\partial\zeta^2} - H\frac{\partial H}{\partial\zeta} - \frac{\partial H}{\partial\tau}.$$

Although the preceding arguments are valid for infinitely extended disks, the results obtained can be applied to disks of finite radius R, if this radius is large in comparison with the distance h. The resistance moments of disks M_0 and M_1, with radii equal to R, will be

$$M_0(\tau) = -2\pi\int_0^R r^2\tau_{z\theta}\,dr = -\frac{\pi\rho R^4}{2}\left(\frac{\nu}{t_0^3}\right)^{1/2}\frac{\partial G(0,\tau)}{\partial\zeta},$$

$$M_1(\tau) = \frac{\pi \rho R^4}{2} \left(\frac{\nu}{t_0^3} \right)^{1/2} \frac{\partial G(\zeta_1, \tau)}{\partial \zeta}. \tag{12}$$

2. It is known that the problem of heat transfer in a viscous incompressible fluid (ν-const) can be solved after the dynamical problem has been solved.

It is easy to show that the unsteady energy equation, written in a cylindrical coordinate system, in the presence of axial symmetry,

$$\rho c_v \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + v_z \frac{\partial T}{\partial z} \right) =$$

$$= \lambda \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} \right) +$$

$$+ A\mu \left\{ 2 \left(\frac{\partial v_r}{\partial r} \right)^2 + 2 \left(\frac{v_r}{r} \right)^2 + 2 \left(\frac{\partial v_z}{\partial z} \right)^2 + \left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right)^2 + \left(\frac{\partial v_\theta}{\partial z} \right)^2 + \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \right)^2 \right\}, \tag{13}$$

where c_v is the specific heat at constant volume; λ is the coefficient of thermal conductivity; A is the thermal equivalent of mechanical energy, admits an exact solution in the form

$$T(r, z, t) = A \frac{\mu \nu}{\lambda t_0} \sum_{k=0}^{n} \left(\frac{r}{\sqrt{\nu t_0}} \right)^{2k} \Theta_{2k}(\zeta, \tau), \tag{14}$$

where n is an arbitrary integer, and the functions $\Theta_{2k}(\zeta, \tau)$ $(k = 0, 1, \ldots, n)$ must satisfy the one-dimensional heat-conduction equation with pere-

with variable coefficients, namely

$$\begin{gathered} \frac{\partial^2 \Theta_0}{\partial \zeta^2} - \sigma \left(\frac{\partial \Theta_0}{\partial \tau} + H \frac{\partial \Theta_0}{\partial \zeta} \right) = -4\Theta_2 - 12F^2, \\ \frac{\partial^2 \Theta_2}{\partial \zeta^2} - \sigma \left(\frac{\partial \Theta_2}{\partial \tau} + 2F\Theta_2 + H \frac{\partial \Theta_2}{\partial \zeta} \right) = -16\Theta_4 - \left(\frac{\partial G}{\partial \zeta} \right)^2 - \left(\frac{\partial F}{\partial \zeta} \right)^2, \\ \frac{\partial^2 \Theta_4}{\partial \zeta^2} - \sigma \left(\frac{\partial \Theta_4}{\partial \tau} + 4F\Theta_4 + H \frac{\partial \Theta_4}{\partial \zeta} \right) = -36\Theta_6, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{gathered} \tag{15}$$

$$\frac{\partial^2\Theta_{2(n-1)}}{\partial\zeta^2} - \sigma\left(\frac{\partial\Theta_{2(n-1)}}{\partial\tau} + 2(n-1)F\Theta_{2(n-1)} + H\frac{\partial\Theta_{2(n-1)}}{\partial\zeta}\right) = -(2n)^2\Theta_{2n},$$

$$\frac{\partial^2\Theta_{2n}}{\partial\zeta^2} - \sigma\left(\frac{\partial\Theta_{2n}}{\partial\tau} + 2nF\Theta_{2n} + H\frac{\partial\Theta_{2n}}{\partial\zeta}\right) = 0;$$

$$\sigma = \mu c_\nu/\lambda$$

is the Prandtl number.

Let us consider several boundary-value problems:

a) the case of a variable temperature along the radius of the disks:

$$\Theta_{2k}(0,\tau) = a_{2k}(\tau), \qquad \Theta_{2k}(\zeta_1,\tau) = b_{2k}(\tau) \quad (k = 0, 1, \dots, n);$$

b) the case of variable heat fluxes along the radius of the disks:

$$\frac{\partial\Theta_{2k}(0,\tau)}{\partial\zeta} = c_{2k}(\tau), \qquad \frac{\partial\Theta_{2k}(\zeta_1,\tau)}{\partial\zeta} = d_{2k}(\tau) \quad (k = 0, 1, \dots, n).$$

Here $a_{2k}, b_{2k}, c_{2k}, d_{2k}$ $(k = 0, 1, \dots, n)$ are prescribed functions of τ. In addition, the initial conditions must be satisfied:

$$\Theta_{2k}(\zeta, 0) = 0 \quad (k = 0, 1, \dots, n).$$

After the heat problem has been solved, the local Nusselt numbers N_0 and N_1 will be given by the formulas:

$$\frac{N_0}{\sqrt{\mathrm{Re}}} = -\frac{\sum\limits_{k=0}^{n}\frac{\partial\Theta_{2k}(0,\tau)}{\partial\zeta}\,\mathrm{Re}^k}{\sum\limits_{k=0}^{n}\Theta_{2k}(0,\tau)\,\mathrm{Re}^k}, \qquad \frac{N_1}{\sqrt{\mathrm{Re}}} = \frac{\sum\limits_{k=0}^{n}\frac{\partial\Theta_{2k}(\zeta_1,\tau)}{\partial\zeta}\,\mathrm{Re}^k}{\sum\limits_{k=1}^{n}\Theta_{2k}(\zeta_1,\tau)\,\mathrm{Re}^k},$$

where

$$\mathrm{Re} = \frac{r^2}{\nu t_0}, \qquad N_0 = \frac{-\lambda\left.\frac{\partial T}{\partial z}\right|_{z=0} r}{\lambda(T - T_0)}, \qquad N_1 = \frac{\lambda\left.\frac{\partial T}{\partial z}\right|_{z=h} r}{\lambda(T - T_0)},$$

T_0 is the characteristic temperature.

The solution obtained is exact in the sense that equations (6), (7), and (15) can be solved, for example, by a numerical method.

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Note: Figure translations are in progress. See original paper for figures.