ON THE STABILITY OF QUASILINEAR SYSTEMS WITH AFTEREFFECT

1958

SovietRxiv

Abstract

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MATHEMATICS

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ON THE STABILITY OF QUASILINEAR SYSTEMS WITH AFTEREFFECT

(Presented by Academician I. G. Petrovskii on 22 VII 1957)

Known stability theorems for ordinary equations ($^{1-4}$) are transferred in this note to systems with aftereffect

$$\frac{dx}{dt} = X[x(t+\vartheta), t] + R[x(t+\vartheta), t] \qquad (-h \le \vartheta \le 0), \tag{1}$$

where x is an element of the Banach space B; X, R are operators on continuous curves $x(\vartheta)$ $(-h \le \vartheta \le 0)$, mapping these curves into B, $t \ge 0$. The space of continuous curves $x(\vartheta)$ with norm $\|x(\cdot)\| = \sup \|x(\vartheta)\|$ for $-h \le \vartheta \le 0$ will be denoted by $B(\cdot)$, and elements of $B(\cdot)$ by $x(\cdot)$.

The stability criteria given below are based on the spectral properties of the linear (bounded) operator X and include certain known results ($^{5-9}$). The linear equation

$$\frac{dx}{dt} = X[x(t+\vartheta)] \qquad (-h \le \vartheta \le 0,\ x \in B,\ \|X\| = L) \tag{2}$$

is equivalent to the "ordinary" equation

$$\left.\frac{dx(t,\cdot)}{dt}\right|_{dt=+0} = Ax(t,\cdot) \qquad (x(t,\cdot) \in B(\cdot)) \tag{3}$$

with the unbounded operator

$$Ax(\cdot) = y(\cdot) = \begin{cases} y(\vartheta) = dx/d\vartheta & \text{for } -h \le \vartheta < 0, \\ y(0) = X[x(\vartheta)]. \end{cases} \tag{4}$$

The investigation of equation (3) is based on results from semigroup theory (^{10}). The stability of the quasilinear equation (1) is studied by the Lyapunov method, whose extension to systems with aftereffect is described in papers (^{12,13}).

Theorem. If the spectrum $\{\lambda_\sigma\}$ of the operator A (4) satisfies the condition

$$\operatorname{Re}\lambda_\sigma \leq -\gamma \qquad (\gamma > 0), \tag{5}$$

then there exists a functional $v[x(\cdot)]$ satisfying the estimates

$$c_1\|x(\cdot)\|^k \leq v[x(\cdot)] \leq c_2\|x(\cdot)\|^k,$$

$$\limsup_{\Delta t\to+0}(\{v[x(t+\Delta t,\cdot)] - v[x(t,\cdot)]\}/\Delta t) \leq -c_3\|x(\cdot)\|^k,$$

$$\left|v[x''(\cdot)] - v[x'(\cdot)]\right| \leq c_4\|x''(\cdot) - x'(\cdot)\| \sup(\|x''(\cdot)\|^{k-1}, \|x'(\cdot)\|^{k-1}), \tag{6}$$

where $k > 0$ is any preassigned integer, and $c_i > 0$ are constants.

Remark. If $B = E_n$, then equation (2) has the form ([5,11])

$$\frac{dx_i}{dt} = \sum_{j=1}^{n}\int_{-h}^{0} x_j(t+\vartheta)\,d\eta_{ij}(\vartheta) \qquad (i = 1, \dots, n) \tag{7}$$

and the spectrum of the operator A is determined by the roots of equation ([5])

$$\left|\int_{-h}^{0} e^{\lambda\vartheta}\,d\eta_{ij}(\vartheta) - \lambda\delta_{ij}\right|_1^n = 0.$$

A second example of equation (2) is the system of integro-differential equations

$$\frac{\partial\varphi_i(\xi,t)}{\partial t} = \sum_{j=1}^{n}\left[\int_a^b K_{ij}(\xi,s)\varphi_j(s,t)\,ds + \int_{-h}^{0}\varphi_j(\xi,t+\vartheta)\,d\eta_{ij}(\vartheta)\right]$$

$$(i = 1, \dots, n;\ a \leq \xi \leq b;\ t \geq 0;\ x = \{\varphi_1(\xi), \dots, \varphi_n(\xi)\}).$$

The proof of the theorem is based on a lemma.

Lemma. *The solutions of equation (3) are asymptotically stable and satisfy the inequality*

$$\|x(x_0(\cdot),t,\cdot)\| \leq N\|x_0(\cdot)\|\exp(-q\gamma t) \qquad (t \geq 0), \tag{8}$$

where q is any chosen number, $0 < q < 1$, for all initial data $x_0(\cdot) \in B(\cdot)$, if condition (5) is satisfied.

A similar criterion for linear equations is given in ([4]); here, however, the theorem from ([4]) is not directly applicable, since in ([4]) the operator A was assumed to be bounded.

We shall briefly describe the proof of the lemma. Consider the operator

$$A_0 x(\cdot) = y(\cdot) = \begin{cases} y(\vartheta) = dx/d\vartheta, & \text{for } -h \le \vartheta < 0, \\ y(0) = X[e^{\gamma\vartheta} x(\vartheta)] + \gamma x(0), \end{cases}$$

whose spectrum $\{\lambda_\sigma^0\}$, by virtue of (5), satisfies the inequality

$$\operatorname{Re} \lambda_\sigma^0 \le 0.$$

Denote by $D[A_0^2]$ the set of elements of $B(\cdot)$ on which the operator A_0^2 is defined. Writing out the resolvent $R(\lambda, A_0)$ of the operator A_0 and repeating, with minor changes, the reasoning of ([10]) (pp. 276–282, 190–194), one can verify that the operator

$$T_0(t) z(\cdot) = \lim_{\omega \to \infty} \frac{1}{2\pi i} \int_{\gamma_1 - i\omega}^{\gamma_1 + i\omega} e^{\lambda t} R(\lambda, A_0) z(\cdot)\, d\lambda \qquad (\gamma_1 > 0,\ t \ge 0) \tag{9}$$

is defined for $z(\cdot) \in D[A_0^2]$ and coincides with the semigroup operator $T_0(t)$ (on $D[A_0^2]$), for which A_0 is an infinitesimal generating operator ([10]).

In other words, $y(z_0(\cdot), t, \cdot) = T(t) z_0(\cdot)$ for $z_0(\cdot) \in D[A_0^2]$, where $y(z_0(\cdot), t, \cdot)$ is a solution of the "equation"

$$\frac{dy(t, \cdot)}{dt} = A_0 y(t, \cdot) \qquad (t \ge 0). \tag{10}$$

Moreover, it is verified that the operator $T_0(t)$ satisfies the inequality

$$\|T_0(t) z_0(\cdot)\| \le \left[N_0 \|z_0(\cdot)\| + N_1 \|A_0 z_0(\cdot)\| + N_2 \left\|A_0^2 z_0(\cdot)\right\|\right] e^{(\gamma_1 + \alpha)t} \tag{11}$$

($t \ge 0$; $z_0(\cdot) \in D[A_0^2]$; $N_i = \text{const}$; α, γ_1 are positive constants which can be chosen arbitrarily small).

For $t = 3h$, the segments of the curves $y(y_0(\cdot), t + \vartheta)$ $(-h \le \vartheta \le 0)$ of the solutions (10) are twice continuously differentiable with respect to ϑ, and, moreover,

$$\frac{d^2 y}{dt^2} = X\left[e^{\gamma\vartheta} \frac{dy(t+\vartheta)}{d\vartheta}\right] + \gamma \left(\frac{dy}{d\vartheta}\right)_{\vartheta = 0}; \tag{12}$$

$$\|y(y_0(\cdot), t, \cdot)\| \leq P_0 \|y_0(\cdot)\| \quad \text{for } 0 \leq t \leq 3h; \tag{13}$$

$$\|y(y_0(\cdot), 3h, \cdot)\| \leq P_1 \|y_0(\cdot)\|; \qquad \left\|A_0^2 y(y_0(\cdot), 3h, \cdot)\right\| \leq P_2 \|y_0(\cdot)\|. \tag{14}$$

for all $y_0(\cdot) \in B(\cdot)$. Therefore $y(y_0(\cdot), 3h, \cdot) \in D[A_0^2]$, and the solutions (10) for $y_0(\cdot) \in B(\cdot)$ satisfy the inequality

$$\|y(y_0(\cdot), t, \cdot)\| \leq P \|y_0(\cdot)\| \exp(\gamma_1 + \alpha)t \quad (P = \text{const},\ t \geq 0). \tag{15}$$

The solutions of equation (2) and of the equation

$$\frac{dy}{dt} = X\left[e^{\gamma\vartheta} y(t+\vartheta)\right] + \gamma y(t),$$

corresponding to the operators A and A_0, are connected, evidently, by the relation

$$x(x_0(\cdot), t) = y(y_0(\cdot), t) \exp(-\gamma t) \quad \text{for } t \geq 0,$$

$$x_0(\vartheta) = y_0(\vartheta) e^{-\gamma\vartheta},$$

whence, in consequence of (15), we draw the conclusion that the lemma is valid.

Now, for the proof of the theorem it is sufficient to consider the functional

$$v[x_0(\cdot)] = \int_0^Q \|x(x_0(\cdot), \xi, \cdot)\|^k \, d\xi + \sup\left[\|x(x_0(\cdot), \xi, \cdot)\|^k \quad \text{for } 0 \leq \xi \leq Q\right]$$

$$\left(Q = \frac{4}{\gamma} \ln 2N \quad \text{for } q = \frac{1}{2} \text{ in formula (8)}\right).$$

The results obtained make it possible to transfer to equations (1) the stability criteria for ordinary quasilinear systems. Assuming that the operator R in (1) satisfies the conditions for the existence of solutions in some neighborhood of zero, $\vartheta(\cdot) \in B(\cdot)$, we give, as examples, two such results.

1. If condition (5) is fulfilled, then one can indicate a constant $a > 0$ such that the solution $x(t, \cdot) \equiv \vartheta(\cdot)$ of the equation

$$\frac{dx}{dt} = X[x(t+\vartheta)] + R[x(t+\vartheta), t] \quad (-h \le \vartheta \le 0)$$

is asymptotically stable, provided only that the inequality

$$\|R[x(\vartheta), t]\| \le a\,\|x(\cdot)\|\,. \tag{16}$$

is satisfied.

This assertion remains valid also in the case when the operator R is defined on curves $x(\vartheta)$ for $\vartheta \in [-h_1, 0]$, where $h_1 > h$. In this case, in inequality (16), on the right-hand side one should write $\|x(\cdot)\| = \sup|x(\vartheta)|$ for $-h_1 \le \vartheta \le 0$.

2. If the spectrum of the operator A_∞

$$A_\infty x(\cdot) = \begin{cases} dx/d\vartheta, & \text{for } -h \leqslant \vartheta < 0, \\ X_\infty[x(\cdot)], & \text{for } \vartheta = 0 \end{cases}$$

satisfies condition (5), and $\|R[x(\cdot), t]\| \leqslant D\|x(\cdot)\|^{1+\beta}$ ($D,\ \beta > 0$—const),

$$\lim_{t\to\infty} \|X[x(\cdot), t] - X_\infty[x(\cdot)]\| = 0$$

uniformly for $\|x(\cdot)\| = 1$ as $t \to \infty$, then the solution $x(t, \cdot) = \theta(\cdot)$ of equation (1) is asymptotically stable, and the characteristic numbers of the solutions

$$\rho = -\overline{\lim_{t\to\infty}} \left(\frac{1}{t} \ln\|x(x(\cdot), t, \cdot)\|\right)$$

satisfy the inequality $\rho \geqslant \gamma$.

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Received
8 IV 1957

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Note: Figure translations are in progress. See original paper for figures.