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Abstract

Full Text

MATHEMATICS

S. N. SLUGIN

ON THE THEORY OF THE METHODS OF NEWTON AND CHAPLYGIN

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1. In an analogue of Newton' s method (1) for the approximate solution of the equation $P(x) = 0$, the algorithm

$$x_{n+1} = x_n - \Gamma_n^{-1} P(x_n), \tag{1}$$

is used, where $\Gamma_x = P'(x_n)$ or $\Gamma_n = P'(x_0)$. Instead of the derivative operators Γ_n in (1), one may take an operator L, having an inverse L^{-1}, "close" to $P'(x)$ in the Banach norm (2) (our notation). We may impose weaker restrictions on the operator L and choose a simpler (not necessarily invertible) operator convenient for computations.

Let X be a space of type (B_K) (3). We shall use the notation:

$$V^n(z) = V[V^{n-1}(z)], \qquad V^0 = I, \qquad I(x) \equiv x, \qquad \Delta U = U(x + \Delta x) - U(x).$$

Theorem 1. Suppose that on the set $G \subset X$: 1) $|\Delta U| \leq V(|\Delta x|)$, where $U = I - LP$; 2) the operator V is monotonically increasing; 3) G contains the bounded sphere (x_0, R):

$$|x - x_0| \leq R = \sum_{k=0}^{\infty} V^k(z), \qquad z = |LP(x_0)|; \tag{2}$$

4) the operator LP is (bk)-continuous; 5) the equation $L(y) = 0$ has the unique zero solution.

Then the algorithm

$$x_{n+1} = x_n - LP(x_n) \qquad (n = 0, 1, 2, ...) \tag{3}$$

converges in the sphere (x_0, R) to the solution x^* of the equation $P(x) = 0$ with rate

$$|x_n - x^*| \le \sum_{k=n}^{\infty} V^k(z). \quad (4)$$

For condition 2) it is sufficient that the operator V be additive; for 4), (o)-continuity of V is sufficient; for 5), additivity and invertibility of the operator L are sufficient. If V is not additive, then in the inequalities (2), (4) one may take $|V_k|$, and, in general, a stronger estimate is obtained than when using the ordinarily employed (in Banach spaces) power $\|V\|^k$ or even the norm $\|V^k\|$ of the iteration.

The operator V is called **Volterra** if $(I-V)^{-1} = \sum_{k=0}^{\infty} V^k$. In this case $R = (I-V)^{-1}(z)$. If V is additive, then $|x_n - x^*| \le V^n(R)$.

2. In what follows X is a K-space ([3]). For practical estimates of a solution it is sometimes necessary to establish an operator analogue of Chaplygin' s theorem—

Lygina [^4] on differential inequalities, i.e., the establishment of conditions sufficient for the inequality $P(x) \ge 0$ to imply $x \ge x^*$, where $P(x^*) = 0$. For an additive invertible operator P this means its positive invertibility, i.e., $P^{-1} > 0$.

Theorem 2 (comparison of additive operators). *If the operators Γ and Λ are additive and invertible, $\Gamma > \Lambda$, $\Gamma^{-1} > 0$, $(I - \Gamma^{-1}\Lambda)^n(x) \overset{(o)}{\to} 0$ $(n \to \infty)$, then $\Lambda^{-1} > 0$.*

Theorem 3 (on operator inequalities). *Suppose that on the set $G \subset X$ the following conditions hold: $P(x_0) > 0$ (or < 0); $\Gamma(\Delta x) \ge \Delta P \ge \Lambda(\Delta x)$ for $\Delta x > 0$; the operators Γ and Λ are additive and positively invertible; the operator $\Gamma^{-1}P$ is monotonically continuous; the set G contains $[x_1, x_0]$ (or $[x_0, x_1]$), where $x_1 = x_0 - \Lambda^{-1}P(x_0)$.*

Then on this interval there exists a unique solution of the equation $P(x) = 0$.

From this, in particular, it is not difficult to obtain the comparison theorem from the paper [^5].

3. Let us construct an algorithm of Chaplygin type on the basis of replacing the operator Γ_n^{-1} in the algorithm of Theorem 1 of the paper [^6], similarly to how this was done above.

Theorem 4. *Suppose that on each $[a,b] \subseteq [x_0, \bar{x}_0]$ there exist additive positively invertible operators $\underline{\Gamma}(a,b)$ and $\bar{\Gamma}(a,b)$ such that throughout $[a,b]$*

$$\underline{\Gamma}(a,b)(x-a) \ge P(x) - P(a), \qquad \bar{\Gamma}(a,b)(b-x) \ge P(b) - P(x).$$

Suppose there exists an additive positively invertible operator $\Gamma \ge \underline{\Gamma}(a,b)$, $\Gamma \ge \bar{\Gamma}(a,b)$ on all $[a,b]$, the operator $\Gamma^{-1}P$ is monotonically continuous, and $P(x_0) \le 0 \le P(\bar{x}_0)$.

Define the algorithm

$$\underline{x}_{n+1} = \underline{x}_n - \underline{L}_n(\underline{z}_n), \qquad \bar{x}_{n+1} = \bar{x}_n - \bar{L}_n(\bar{z}_n), \tag{5}$$

where $P(\underline{x}_n) \leq \underline{z}_n \leq 0 \leq \bar{z}_n \leq P(\bar{x}_n)$, the operators L_n are positive and homogeneous,

$$\underline{\Gamma}_n \underline{L}_n \leq I, \qquad \bar{\Gamma}_n \bar{L}_n \leq I, \qquad \underline{\Gamma}_n = \underline{\Gamma}(\underline{x}_n, \bar{x}_n), \quad \bar{\Gamma}_n = \bar{\Gamma}(\underline{x}_n, \bar{x}_n).$$

Then the algorithm determines sequences x_n satisfying the inequalities

$$\underline{x}_n \leq \underline{x}_{n+1} \leq \underline{x} \leq \bar{x} \leq \bar{x}_{n+1} \leq \bar{x}_n,$$

where $\underline{x}, \bar{x}$ are the least and greatest solutions on $[x_0, \bar{x}_0]$ of the equation $P(x) = 0$.

For uniqueness of the solution it is sufficient that there exist an additive positively invertible operator Λ such that $\Delta P \geq \Lambda(\Delta x)$ for $\Delta x > 0$.

If it is known in advance that the solution is unique, then the condition $x_0 \leq \bar{x}_0$ need not be imposed. Instead of the conditions imposed on L_n, it is sufficient to require that $\underline{\Gamma}_n \underline{L}_n(\underline{z}_n) \geq \underline{z}_n$, $\bar{\Gamma}_n \bar{L}_n(\bar{z}_n) \leq \bar{z}_n$. Under some additional assumptions concerning $L_n(z_n)$, the algorithm converges to the solution. Instead of continuity of the operator $\Gamma^{-1}P$, one may require continuity of the operators Γ and P. This remark also applies to Theorem 3.

Putting $\Gamma_n = \bar{\Gamma}_n$, $L_n = \Gamma_n^{-1}$, we obtain the algorithms of the paper [^7].

Theorem 5. *If, under the conditions of Theorem 4, the operators $I - \Gamma_n$ are positive Volterra operators, then one may take as L_n*

$$\sum_{k=0}^{m} (I - \Gamma_n)^k$$

where m is arbitrary.

Hence, for $z_n = P(x_n)$, acting in the space of collections of derivatives, one can, in particular, obtain (concrete) algorithms of work[8]. They occupy an intermediate place between the algorithms

$$x_{n+1} = x_n - P(x_n)$$

and the more complicated, but more rapidly convergent,

$$x_{n+1} = x_n - \Gamma_n^{-1} P(x_n).$$

4. The additivity and positive invertibility condition on the operator Γ', usually used in analogues of Chaplygin' s method[6,7], can be replaced by another one.

Theorem 6. *Let the following conditions be fulfilled on the set* $G \subset X$*:* $P(x_0) > 0$ *(or* < 0*),* $\Gamma(\Delta x) \geq \Delta P$ *for* $\Delta x > 0$*; there exists an operator* L *such that* $L > 0$*,* $\Gamma L \leq I$ *(or* $L(y) < 0 \leq \Gamma L(y) - y$ *for* $y < 0$*); the equation* $L(y) = 0$ *has the unique zero solution; the operator* LP *is monotonically continuous;* $\Delta U \leq \Delta V$ *for* $\Delta x > 0$*, where* $U = I - LP$*; the operator* V *is monotonically increasing; the set* G *contains the bounded interval*

$$[x_0 - R, x_0] \quad (\text{or } [x_0, x_0 + R]),$$

where

$$R = \sum_{k=0}^{\infty} V^k(z), \qquad z = |LP(x_0)|.$$

Then the equation $P(x) = 0$ *has on this interval a solution* x^**, which can be obtained by algorithm (3), and moreover* $x_n \searrow x^*$ *(or* $x_n \nearrow x^*$*) with rate*

$$|x_n - x^*| \leq \sum_{k=n}^{\infty} V^k(z).$$

A consequence of this is Theorem 3 under the additional condition: the operator $V = I - \Gamma^{-1}\Lambda$ is Volterra. The theorem can be modified by replacing algorithm (3) with an algorithm of the form (5).

Kazan State
University

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CITED LITERATURE

[1] L. V. Kantorovich, *DAN*, **76**, No. 1, 17 (1951).
[2] B. A. Vertgeim, *Uspekhi Mat. Nauk*, **12**, issue 1 (73), 166 (1957).
[3] L. V. Kantorovich, B. Z. Vulikh, A. G. Pinsker, *Functional Analysis in Semi-Ordered Spaces*, Moscow-Leningrad, 1950.
[4] S. A. Chaplygin, *A New Method of Approximate Integration of Differential Equations*, Moscow-Leningrad, 1950.
[5] N. V. Azbelev, *DAN*, **89**, No. 4, 589 (1953).
[6] S. N. Slugin, *DAN*, **103**, No. 4, 565 (1955).
[7] A. N. Baluev, *Vestn. LGU*, No. 13, issue 3, 27 (1956).
[8] G. A. Artemov, *DAN*, **101**, No. 2, 197 (1955).