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Abstract

Full Text

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THE GENUS OF A FIBER SPACE

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This note introduces the concept of the genus of a fiber space. This concept includes the notion, introduced earlier by the author ⁽¹⁾ (and, in special cases, by M. A. Krasnosel'skii ⁽²⁾ and Yan Chzhun-dao ⁽³⁾), of the genus of a topological space with respect to a finite group of transformations acting in it without fixed points. On the other hand, it encompasses the concept of the category of a topological space in the sense of Lusternik and Schnirelmann.

All fiber spaces considered are assumed to be locally trivial, and all topological spaces—normal. The category of a topological space B is defined as the least cardinality of an open covering of the space B consisting of sets contractible in B , and is denoted $\text{cat } B$. By m, n arbitrary cardinal numbers (not necessarily finite) are denoted.

Definition 1. The **genus of a fiber space** (E, B, F, p) (denoted $g(E, B, F, p)$) is the least cardinality of an open covering of the base B consisting of sets over each of which there exists a cross-section.

1. Behavior of the genus under mappings of fiber spaces. The following obvious propositions hold.

Proposition 1. Let (E, B, F, p) and (E', B, F', p') be fiber spaces, and let f be a mapping of the space E into E' satisfying the condition $p'f = p$. Then $g(E', B, F', p') \leq g(E, B, F, p)$.

Proposition 2. If the principal fiber space (E, B, G, p) can be admissibly mapped into the principal fiber space (E_1, B_1, G, p_1) , then $g(E, B, G, p) \leq g(E_1, B_1, G, p_1)$.

2. Relation of the genus of a fiber space to the category of its base.

Theorem 1. $g(E, B, F, p) \leq \text{cat } B$. If the space E is contractible, then $g(E, B, F, p) = \text{cat } B$.

Proof. From the covering homotopy theorem it follows that over a set contractible in the base of a fiber space there exists a cross-section. In the case, however, when the total space E of the fibration is contractible, the converse is also true: if over a set A there exists a cross-section φ , then the set A is contractible in the base B . For the proof it is enough to contract the set $\varphi(A)$ to a point in the space E and “project” this deformation to the base B .

3. Relation of the genus of a fiber space to dimension:

Theorem 2. Let (E, B, F, p) be a fiber space whose fiber is aspherical in dimensions less than s ($\pi_0(F) = \dots = \pi_{s-1}(F) = 0$), and let the base be a k -dimensional polyhedron. Then

$$g(E, B, F, p) < \frac{k+1}{s+1} + 1.$$

Proof. Let K^* be the barycentric subdivision of some triangulation K of the polyhedron B . Denote by \mathfrak{A}_i the set of those vertices of the triangulation K^* which are centers of simplices of the triangulation K having dimension t lying in the range

$$(s+1)(i-1) \leq t < (s+1)i.$$

Consider the sets A_i , which are the unions of the stars of those vertices of the complex K^* that belong to the set \mathfrak{A}_i . Over each of the sets A_i there exists a cross-section, since each of these sets can be deformed into a subcomplex of the triangulation K^* having dimension $\leq s$ (namely, into the subcomplex consisting of the simplices of the triangulation K^* all of whose vertices belong to the set \mathfrak{A}_i). Since the sets A_i form a covering of the base and among them there are fewer than

$$\frac{k+1}{s+1} + 1$$

nonempty ones, the theorem is proved.

Corollary 1. Let B be a k -dimensional polyhedron, aspherical in dimensions less than s . Then

$$\text{cat } B < \frac{k+1}{s} + 1.$$

Corollary 1 was previously obtained by Grossman ⁽⁴⁾.

Theorem 3. Let (E, B, F, p) be a fiber space whose base is a polyhedron, and let $\varphi : B \rightarrow M$ be a mapping of its base B into an m -dimensional bicompectum M . Then there exists a point $x \in M$ such that

$$g(p^{-1}\varphi^{-1}(x), \varphi^{-1}(x), F, p) \geq \frac{1}{m+1} g(E, B, F, p).$$

4. Homological estimate of the genus of a fiber space

Theorem 4. Suppose that in the cohomology ring $H(B; A)$ of the base B of the fiber space (E, B, F, p) there are n elements x_1, \dots, x_n satisfying the conditions

$$p^*x_1 = \dots = p^*x_n = 0, \quad x_1x_2 \dots x_n \neq 0.$$

Then

$$g(E, B, F, p) \geq n + 1.$$

The proof of this theorem is entirely analogous to the proof of Theorem 4 of note ⁽¹⁾.

5. Universal principal fiber spaces of genus n

Let G be a topological group. Denote by $\Pi(G)$ the pyramid over the group G , i.e. the space obtained from the product of the group G with the interval $[0, 1]$ by identifying the lower base $G \times 0$ to a single point O . By $\Pi'(G)$ we shall mean the set $\Pi(G) \setminus G \times 1$. Consider now the space

$$E_n(G) = (\Pi(G))^n \setminus (\Pi'(G))^n.$$

In the space $E_n(G)$ the group G acts naturally (coordinatewise) without fixed points. The fibration of the space $E_n(G)$ into trajectories of the group G determines a principal fiber space

$$(E_n(G), B_n(G), G, p_n).$$

It is not hard to see that

$$g(E_n(G), B_n(G), G, p_n) \leq n.$$

Theorem 5. *Any principal fiber space (E, B, G, p) having genus n can be admissibly mapped into the fiber space*

$$(E_n(G), B_n(G), G, p_n).$$

Proof. Let $\beta = \{B_1, \dots, B_n\}$ be an open covering of the space B by sets over each of which there exists a cross-section, and let

$$\varphi_i : B_i \rightarrow E$$

be a cross-section over the set B_i . Construct a system of real functions f_1, \dots, f_n on the space B having the following properties: a) $0 \leq f_i \leq 1$; b) $f_i = 0$ outside the set B_i ; c) at each point $x \in B$, for some i the function $f_i(x) = 1$.

For each i ($1 \leq i \leq n$) define a mapping

$$\Phi_i : E \rightarrow \Pi(G)$$

commuting with the transformations of the group G , by means of the following construction. To each point $x \in p^{-1}(B_i)$ assign the point $(g, t) \in \Pi(G)$, where the element $g \in G$ is determined by the relation

$$x = g\varphi_i p(x),$$

and the number t by the formula

$$t = f_i p(x).$$

To points of the set $E \setminus p^{-1}(B_i)$ assign the point $O \in \Pi(G)$. The totality of mappings Φ_1, \dots, Φ_n determines

has the map $\Phi : E \rightarrow (\Pi(G))^n$. It is easy to verify that $\Phi(E) \subset E_n(G)$, and, consequently, we have constructed an admissible map of the fiber space (E, B, G, p) into the fiber space $(E_n(G), B_n(G), G, p_n)$.

Modifying the proof of Theorem 5, one obtains the following assertion.

Theorem 6. *Principal fiber spaces (E, B, G, p) with base of weight $\leq \tau$ are in one-to-one correspondence with the homotopy classes of maps of the base B into the space $B_\tau(G)$ (i.e. $B_\tau(G)$ is a classifying space for fiber spaces with base of weight $\leq \tau$).*

In this theorem τ denotes an infinite cardinal number.

6. Estimates for the genus of a principal fiber space. In what follows we shall consider only fiber spaces with triangulable base, although the results obtained are valid under more general assumptions.

First of all note that the fiber space $(E_n(G), B_n(G), G, p_n)$, where n is a natural number, is naturally embedded in the fiber space $(E_\omega(G), B_\omega(G), G, p_\omega)$, where ω is countable cardinality. Denote by φ_n the embedding of $B_n(G)$ in $B_\omega(G)$. The fiber space (E, B, G, p) can be specified by means of a characteristic map $\varphi : B \rightarrow B_\omega(G)$. From Theorem 5 it follows immediately:

Proposition 3. *$g(E, B, G, p) \leq n$ if and only if the characteristic map $\varphi : B \rightarrow B_\omega(G)$ is homotopic to a map carrying B into $B_n(G)$.*

It follows at once from this proposition that, if $g(E, B, G, p) = n$, then for any coefficient group A the kernel of the map

$$\varphi^* : H(B_\omega(G); A) \rightarrow H(B; A)$$

contains the kernel of the map

$$\varphi_n^* : H(B_\omega(G); A) \rightarrow H(B_n(G); A).$$

This gives a lower estimate for the genus. But other estimates can also be obtained from Proposition 3.

First observe that

$$\pi_i(B_\omega(G), B_n(G)) \approx \pi_i(E_\omega(G), E_n(G)) \approx \pi_{i-1}(E_n(G)).$$

Suppose that the group G is aspherical in dimensions less than s . By the Hurewicz theorem,

$$\pi_i(B_\omega(G), B_n(G)) = 0 \quad \text{for } i < (s+1)n,$$

$$\pi_{(s+1)n}(B_\omega(G), B_n(G)) = H_{(s+1)n-1}(E_n(G)) = H_s(G) \otimes \cdots \otimes H_s(G)$$

(with n factors; for $s = 0$, $H_0(G)$ denotes the zero-dimensional reduced homology group of the space G). Hence it follows that a map of any space of dimension $< (s+1)n$ into the space $B_\omega(G)$ is homotopic to a map whose image is contained

in the set $B_n(G) \subset B_\omega(G)$. Thus a new proof of Theorem 2 for principal fiber spaces is obtained.

Let G be a topological group whose first nontrivial homotopy group has dimension s . Denote by $\xi(n, G)$ the characteristic class of the pair $(B_\omega(G), B_n(G))$ (by definition,

$$\xi(n, G) \in H^{(s+1)n}(B_\omega(G); \pi_{(s+1)n}(B_\omega(G), B_n(G))) \approx H^{(s+1)n}(B_\omega(G), H_s(G) \otimes \cdots \otimes H_s(G)),$$

where in the case $s = 0$ the homology, generally speaking, is considered with local coefficients).

Theorem 7. *Let (E, B, G, p) be a fiber space whose base is an $(s + 1)n$ -dimensional polyhedron, and let $\varphi : B \rightarrow B_\omega(G)$ be the characteristic map of this fiber space. In order that the fiber space (E, B, G, p) have genus $n + 1$, it is necessary and sufficient that the cohomology class $\varphi^*\xi(n, G)$ be nonzero.*

For the proof it is enough to apply the theorem of Pontryagin-Postnikov ([5], p. 133), where this theorem is proved correctly but formulated incorrectly.

In the case when $s = 0$, i.e. the group G is not connected, Theorem 6 can be somewhat modified.

Denote by U the connected component of the identity of the group G , and by N the group $G/U = \pi_0(G)$. To the principal fiber space (E, B, G, p) there naturally corresponds the fiber space $(E/U, B, N, p_1)$. We shall denote the characteristic map of the fibration (E, B, G, p) by φ , the characteristic map of the fibration $(E/U, B, N, p_1)$ by φ_1 , and the natural map $B_\omega(N)$ to $B_\omega(G)$ by ρ . Let the dimension of the polyhedron B be denoted by n .

Theorem 8. The following four conditions are equivalent:

- a) $g(E, B, G, p) = n + 1$;
- b) $\varphi^*\xi(n, G) \neq 0$;
- c) $g(E/U, B, N, p_1) = n + 1$;
- d) $\varphi_1^*\xi(n, N) \neq 0$.

Indeed, by Theorem 6, condition a) is equivalent to condition b), and condition c) to condition d). The equivalence of conditions b) and d) follows from the relation $\varphi^* = \varphi_1^*\rho^*$ and from the fact that the map

$$\rho_* : \pi_n(B_\omega(N), B_n(N)) \rightarrow \pi_n(B_\omega(G), B_n(G))$$

is an isomorphism.

Corollary 2. In order that the category of an n -dimensional polyhedron be equal to $n + 1$, it is necessary and sufficient that its one-dimensional homotopy category be equal to $n + 1$.

Corollary 3. If an n -dimensional manifold has a fundamental group of order 2, then its category is equal to $n + 1$ if and only if its length is equal to n .

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Note: Figure translations are in progress. See original paper for figures.

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