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# THE MEMBERSHIP PROBLEM FOR DIRECT PRODUCTS OF GROUPS

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**Abstract**

**Full Text**

**MATHEMATICS**

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## **THE MEMBERSHIP PROBLEM FOR DIRECT PRODUCTS OF GROUPS**

*(Presented by Academician I. M. Vinogradov on 17 XII 1957)*

By the **strong membership problem** for a group  $\mathfrak{A}$  we shall mean the following problem:

Does there exist an algorithm which makes it possible to determine, for every word of the group  $\mathfrak{A}$  and for every subgroup generated by a finite number of words of the group  $\mathfrak{A}$ , whether this word belongs to the subgroup?

By the **weak membership problem** for a group  $\mathfrak{A}$  we shall mean the problem:

Do there exist algorithms for every subgroup generated by a finite number of words of the group  $\mathfrak{A}$ , making it possible to determine, for every word of the group  $\mathfrak{A}$ , whether it belongs to the subgroup?

In the case of the existence of an algorithm (algorithms) we shall say that the **membership problem for the group  $\mathfrak{A}$  is solvable**. In the contrary case, i.e. if there is no such algorithm (algorithms), we shall say that **for the group  $\mathfrak{A}$  the membership problem is unsolvable**.

From the solvability of the strong membership problem there follows the solvability of the weak membership problem.

The **identity problem** for a group  $\mathfrak{A}$  can be formulated as follows:

Does there exist an algorithm which makes it possible to determine, for every word of the group  $\mathfrak{A}$ , whether it belongs to the unit subgroup of the group  $\mathfrak{A}$ ? From the solvability of the weak membership problem for the group  $\mathfrak{A}$  there follows the solvability of the identity problem for it.

We shall consider finitely presented groups, i.e. groups given by a finite number of generators and a finite number of defining relations.

It is known that for a free group the strong membership problem is solvable <sup>(1)</sup>.

The present article contains some results concerning the solvability of the membership problem, both weak and strong, for direct products of groups. First of all the question arises: does a direct product preserve the solvability of the strong or weak membership problem for a group? This question is answered in the negative, as follows from Theorem 1.

**Theorem 1.** *The weak membership problem for the direct product of two free groups, each of which is given by two generators, is unsolvable.*

Let  $\mathfrak{A}$  be a finitely presented group with an unsolvable identity problem (the existence of such a group was proved by P. S. Novikov <sup>(2)</sup>), given by  $k$  generators  $a_1, \dots, a_k$ , among which are contained all their inverses, and by  $n$  defining relations  $A_i = 1$ , where  $i = 1, \dots, n$ .

Denote by  $\mathfrak{A}_0$  the free group with generators  $a_1, \dots, a_k$ . Consider the subgroup of the group  $\mathfrak{A}_0$  containing such and only such words which have the form

$$U = X_1^{-1} A_{j_1} X_1 \dots X_k^{-1} A_{j_k} X_k, \quad (1)$$

where  $X_q$  are words composed of the letters  $a_1, \dots, a_k$ , and  $1 \leq j_q \leq n$ .

In order for a word to be equal to the identity in the group  $\mathfrak{A}$ , it is necessary and sufficient that it have the form (1).

By  $\mathfrak{A}_0^+$  we denote the free group with generators  $a_1^+, \dots, a_k^+$ , numbered in such an order that the inverse elements for the generators of the groups  $\mathfrak{A}_0$  and  $\mathfrak{A}_0^+$  with the same numbers have the same numbers.

Denote by  $G$  the subgroup of the direct product  $\mathfrak{A}_0 \times \mathfrak{A}_0^+$  generated by a finite number of words  $A_1, \dots, A_n, a_1^{-1} a_1^+, a_2^{-1} a_2^+, \dots, a_k^{-1} a_k^+$ . Every word of the form (1) belongs to the group  $G$ , and every word of the group  $G$  that does not contain generators of the group  $\mathfrak{A}_0^+$  has the form (1).

1) Let

$$U = X_1^{-1} A_{j_1} X_1 X_2^{-1} A_{j_2} X_2 \dots X_k^{-1} A_{j_k} X_k.$$

It is enough to show that every word  $U' = X^{-1} A_{jX}$ , where  $j = 1, \dots, n$ , belongs to the group  $G$ . Let  $X = a_{i_1} \dots a_{i_n}$ ; then  $X^{-1} = a_{i_n}^{-1} \dots a_{i_1}^{-1}$ . Consider the word  $Y = a_{i_n}^+ \dots a_{i_1}^+$ ; then  $Y^{-1} = a_{i_1}^{+1} \dots a_{i_n}^{+1}$ ,

$$X^{-1} A_{jX} = X^{-1} Y A_j Y^{-1} X = a_{i_n}^{-1} a_{i_n}^+ \dots a_{i_1}^{-1} a_{i_1}^+ A_j a_{i_1} a_{i_1}^{+1} \dots a_{i_n} a_{i_n}^{+1},$$

i.e.  $X^{-1} A_{jX}$  belongs to the group  $G$ , and consequently  $U$  also belongs to the group  $G$ .

2) Let

$$U = Y_1 A_{j_1} Y_2 A_{j_2} \dots A_{j_k} Y_{k+1},$$

where  $Y_i$  is composed of the generators  $a_1^{-1} a_1^+, a_2^{-1} a_2^+, \dots, a_k^{-1} a_k^+$ . Write the word  $U$  in the form

$$U = Z_1 A_{j_1} Z_1^{-1} Z_2 A_{j_2} Z_2^{-1} \dots Z_k A_{j_k} Z_k Z_{k+1},$$

where  $Z_1 \equiv Y_1$ ,  $Z_i \equiv Y_{i-1} Y_i$ ,  $i = 1, \dots, k+1$ . Obviously,  $Z_{k+1} = 1$ . After canceling the letters  $a_1^+, \dots, a_k^+$ , we obtain

$$U = X_1^{-1} A_{j_1} X_1 \dots X_n^{-1} A_{j_n} X_n.$$

Hence the following proposition follows: from the solvability of the weak membership problem for the group  $\mathfrak{A}_0 \times \mathfrak{A}_0^+$  follows the solvability of the identity problem for the group  $\mathfrak{A}$ .

Using this proposition and the fact that every free group with any finite number of generators is a subgroup of a free group with two generators, we obtain the result formulated in the theorem.

Thus, the direct product preserves the solvability of neither the strong nor the weak membership problem; nevertheless, one can single out some classes of groups for which the direct product with any group having solvable strong (weak) membership problem preserves the solvability of the strong (weak) membership problem. Such a class turned out, in particular, to be the class of abelian groups.

**Theorem 2.** *The membership problem for the direct product of an abelian group with a group having solvable strong (weak) membership problem is solvable in the strong (weak) sense.*

Because it is impossible to give the complete proof within the bounds of the present article, we shall restrict ourselves only to its outline.

Let  $\mathfrak{A}$  be a cyclic group given by a generator  $a$ ; let  $\mathfrak{B}$  be a group with solvable membership problem. Let  $G$  be a subgroup of the direct product, given by a finite number of generators. Obviously, all generators of the subgroup can be written in the following form:

$$a^{m_i}; a^{n_j}B_j; B'_s,$$

where all  $m_i$  or all  $n_j$  may be equal to zero, and  $B_j$  and  $B'_s$  are words of the group  $\mathfrak{B}$ .

Let  $q$  be the greatest common divisor of the numbers  $m_i$  and  $n_j$  (if all  $m_i$  and  $n_j$  are equal to zero, then the group  $G$  is a subgroup of the group  $\mathfrak{B}$ , for which the membership problem is solvable), and let  $d$  be the greatest common divisor of the numbers  $m_i$  (if all  $m_i$  are equal to zero, then  $d = 0$ ),  $d = rq$ . Then the generators of the subgroup  $G$  can be written in the form

$$a^d; a^qB_0; B'_s,$$

where  $B_0$  is a fixed word from the group  $\mathfrak{B}$ , and among the  $B'_s$ , besides  $B'_s$ , there are also some words from  $G$  which are products of generators of the group  $\mathfrak{B}$ .

Let  $G^*$  be the subgroup of the group  $G$  consisting of those and only those words which, in at least one of their representations, do not contain the generator  $a$  of the group  $\mathfrak{A}$ . The question of whether an arbitrary word  $X$  belongs to the group  $G$

is equivalent to the question of the membership in the group  $G^*$  of some word  $Y$  from the group  $\mathfrak{B}$ , found from the word  $X$ .

For the case when  $\mathfrak{A}$  is a finite group, it has been possible to show that the group  $G^*$  can be given by a finite number of generators. And since in this case  $G^*$  is a subgroup of a group having a decidable membership problem, the membership problem will also be decidable for the direct product  $\mathfrak{A} \times \mathfrak{B}$ .

If, however,  $\mathfrak{A}$  is an infinite group, two cases are considered: 1)  $r \neq 0$  and 2)  $r = 0$ , where  $r = d/q$ .

In the first case it is proved that  $G^*$  can be given by a finite number of generators.

For the second case the following proposition is proved:

If for the group  $K$  and for every word from the direct product  $\mathfrak{A} \times \mathfrak{B}$  one can recognize whether it belongs to the group  $K$ , then also for the group  $K^*$  and for every word from the direct product  $\mathfrak{A} \times \mathfrak{B}$  one can recognize whether it belongs to the group  $K^*$ , where  $K$  is the subgroup of the group with generators  $a^{r^q}; a^q B_0; B_s''$  ( $r \neq 0$ ), consisting of such and only such words that do not contain the generator  $a$  of the group  $\mathfrak{A}$ , and  $K^*$  is the subgroup of the group with generators  $a^q B_0; B_s''$ , consisting of such and only such words that do not contain the generator  $a$  of the group  $\mathfrak{A}$ . Thus, the second case is reduced to the first.

For the group  $G^*$  one can carry out the same reasoning as in the case when  $\mathfrak{A}$  is a finite cyclic group; whence we also obtain the decidability of the membership problem for the direct product of a cyclic group and a group with decidable membership problem.

Taking into account the circumstance that an abelian group may be regarded as a direct product of cyclic groups, we obtain the result formulated in Theorem 2.

If the class of abelian groups is somewhat enlarged, then the following theorem turns out to be true.

**Theorem 3.** *For the direct product of a group with decidable weak membership problem and a group with decidable strong membership problem, every subgroup of which can be given by a finite number of generators, one cannot prove the undecidability of the weak membership problem.*

Let  $\mathfrak{A}$  be a group with decidable strong membership problem and with the property indicated in the statement of the theorem, and let  $\mathfrak{B}$  be a group with decidable weak membership problem. Let  $G$  be a subgroup of the group  $\mathfrak{A} \times \mathfrak{B}$  with generators  $C_1, \dots, C_k; C_i = A_i B_i$ , where  $A_i \in \mathfrak{A}$ ,  $B_i \in \mathfrak{B}$  ( $i = 1, \dots, k$ ). Let  $X = AB \in G$ , where  $A \in \mathfrak{A}$ ,  $B \in \mathfrak{B}$ . Then  $AB = C_{i_1} \dots C_{i_l} = A_{i_1} B_{i_1} \dots A_{i_l} B_{i_l}$ . Consequently,  $B$  is representable as a product of right-hand parts of the words  $C_i$ . The same is true also of  $B^{-1}$ :  $B^{-1} = B_{j_1} \dots B_{j_l}$ . Denote the product of the corresponding left-hand parts of  $C_i$  by  $A^*$ , i.e.  $A^* = A_{j_1} \dots A_{j_l}$ . Then  $ABB^{-1}A^* \in G$ , since  $B^{-1}A^* \in G$ . Consequently,  $AA^* \in G$ , and conversely, if  $AA^* \in G$ , then  $AB \in G$ .

Thus, the problem is reduced to being able to determine, for words of the form

$AA^*$ , whether they belong to the subgroup of the group  $G$ , which can be given by a finite number of generators.

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*Note: Figure translations are in progress. See original paper for figures.*

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