On the Connection of the Method of Nearby Systems in Special Linear Topological Spaces with Certain Questions in the Perturbation Theory of Linear Operators in Banach Spaces

1958

SovietRxiv

Abstract

Full Text

Mathematics

K. M. Fishman

On the Connection of the Method of Nearby Systems in Special Linear Topological Spaces with Certain Questions in the Perturbation Theory of Linear Operators in Banach Spaces

(Presented by Academician V. I. Smirnov on 28 IV 1958)

In the present note we show the possibility of transferring a number of results of the perturbation theory of linear operators in Banach spaces ([1]) to certain linear topological spaces. In particular, from an abstract point of view the method of proximity of M. A. Evgrafov in analytic space (([2]), Ch. IV, §§ 3, 4) is elucidated, which makes it possible not only to strengthen known results, but also to broaden the range of their applicability.

1. Let $\mathcal{B}_r$ $(a < r < b)$ be a family of Banach spaces satisfying the conditions: 1) $\mathcal{B}_r$ is an everywhere dense linear manifold in $\mathcal{B}_{r'}$, with respect to the norm $\|\cdot\|_{r'}$ in $\mathcal{B}_{r'}$, for $r' < r$; 2) $\|f\|_{r'} \leq \|f\|_r$ for $f \in \mathcal{B}_r$ and $r' < r$. Let $\widetilde{\mathcal{B}}_{\tilde{r}}$ $(\tilde{a} < \tilde{r} < \tilde{b}$; $\tilde{r}$ is a continuous monotonically increasing function of $r)$ be another family of Banach spaces with the same properties as $\mathcal{B}_r$. By $\mathfrak{A}_r$ we denote the complete linear topological space consisting of the elements

$$\prod_{r'<r} \mathcal{B}_{r'}$$

with linear operations induced from $\mathcal{B}_{r'}$, $(r' < r)$, and with convergence $f_n \underset{r}{\to} f$ equivalent to the convergence $\|f_n - f\|_{r'} \to 0$ for all $r' < r$. Analogously, starting from the family $\widetilde{\mathcal{B}}_{\tilde{r}}$, we construct the spaces $\widetilde{\mathfrak{A}}_{\tilde{r}}$.* In what follows the asterisk sign for spaces will denote passage to conjugate spaces, and for operators passage to conjugate operators. It is obvious that

$$\mathfrak{A}_r^* = \sum_{r'<r} \mathcal{B}_{r'}^*.$$

Consider a distributive operator A, defined on a linear manifold $D_A(\mathcal{B}_r) \subset \mathcal{B}_r$ $(a < r < b)$, with range $R_A(\widetilde{\mathcal{B}}_{\tilde{r}}) \subset \widetilde{\mathcal{B}}_{\tilde{r}}$. It is assumed that $D_A(\mathcal{B}_r) \subset D_A(\mathcal{B}_{r'})$ for $r' < r$ and that the result of the action of the operator A does not depend on r. The operator A induces in each space $\mathfrak{A}_r$ a distributive operator with domain of definition

$$D_A(\mathfrak{A}_r) = \prod_{r'<r} D_A(\mathcal{B}_{r'})$$

and range

$$R_A(\widetilde{\mathfrak{A}}_{\tilde{r}}) = \prod_{r'<r} R_A(\widetilde{\mathcal{B}}_{\tilde{r}'}).$$

Theorem 1. If A is a Φ-operator** from $\mathcal{B}_r$ into $\widetilde{\mathcal{B}}_{\tilde{r}}$ for all $a<r<b$, then it is also a Φ-operator from $\mathfrak{A}_r$ into $\widetilde{\mathfrak{A}}_{\tilde{r}}$ for all $r \in (a,b)$.

---

* We shall denote the norm in $\widetilde{\mathcal{B}}_{\tilde{r}}$ by $\|\cdot\|_{\tilde{r}}$, and by $\tilde{f}_n \underset{\tilde{r}}{\to} \tilde{f}$ convergence in $\widetilde{\mathfrak{A}}_{\tilde{r}}$.

** The concept of a Φ-operator, introduced in (1) (p. 52) for Banach spaces, includes only topological properties and therefore may also be introduced for operators in linear topological spaces.

Proof. The operator A from $\mathfrak{A}_r$ into $\widetilde{\mathfrak{A}}_r$ is, obviously, closed. Denote, following (1) (p. 50), by $\alpha_A(\mathfrak{B}_r)$ ($\alpha_A(\mathfrak{A}_r)$) the dimension of the null subspace $\mathfrak{z}_A(\mathfrak{B}_r)$ ($\mathfrak{z}_A(\mathfrak{A}_r)$), and by $\beta_A(\widetilde{\mathfrak{B}}^*_r)$ ($\beta_A(\widetilde{\mathfrak{A}}^*_r)$) the dimension of the defect subspace $\mathfrak{z}^*_A(\widetilde{\mathfrak{B}}^*_r)$ ($\mathfrak{z}^*_A(\widetilde{\mathfrak{A}}^*_r)$) of the operator A, acting from $\mathfrak{B}_r$ into $\widetilde{\mathfrak{B}}_r$ (from $\mathfrak{A}_r$ into $\widetilde{\mathfrak{A}}_r$). From the assumption made and from the fact that $\mathfrak{z}_A(\mathfrak{B}_{r'}) \supset \mathfrak{z}_A(\mathfrak{B}_r)$ for $r'<r$, it follows that $\alpha_A(\mathfrak{B}_r)$ ($a<r<b$) is a nondecreasing function taking integer nonnegative values. Consequently, for every r there exists such a $\delta(r)>0$ (we choose $\delta(r)$ maximal possible) that $\alpha_A(\mathfrak{B}_{r'}) = \text{const}$ for $r' \in (r-\delta(r), r)$. Then $\mathfrak{z}_A(\mathfrak{B}_{r'})$ is one and the same for all $r' \in (r-\delta(r), r)$, and therefore $\mathfrak{z}_A(\mathfrak{A}_r) = \mathfrak{z}_A(\mathfrak{B}_{r'})$ ($r-\delta(r)<r'<r$). From the inclusion $\mathfrak{z}^*_A(\widetilde{\mathfrak{B}}^*_{r'}) \subset \mathfrak{z}^*_A(\widetilde{\mathfrak{B}}^*_r)$ ($r'<r$) it follows that $\beta_A(\widetilde{\mathfrak{B}}^*_r)$ is a nonincreasing integer-valued function of r. There therefore exists such an $\eta(r)>0$ that $\beta_A(\widetilde{\mathfrak{B}}^*_{r'}) = \text{const}$ for $r' \in (r-\eta(r), r)$ and, consequently, $\mathfrak{z}^*_A(\widetilde{\mathfrak{B}}^*_{r'})$ is one and the same for all $r' \in (r-\eta(r), r)$. Denote by $\{\widetilde{\Phi}_i\}_1^{m_r}$ ($m_r = \beta_A(\widetilde{\mathfrak{B}}^*_{r'})$) a basis in the spaces $\mathfrak{z}^*_A(\widetilde{\mathfrak{B}}^*_{r'})$ for $r' \in (r-\eta(r), r)$. If $\tilde{h} \in R_A(\widetilde{\mathfrak{A}}_r) \subset R_A(\widetilde{\mathfrak{B}}_{r'})$, then $\widetilde{\Phi}_i(\tilde{h}) = 0$ ($i=1,2,\dots,m_r$). Conversely, let $\tilde{h} \in \widetilde{\mathfrak{A}}_r$, $\widetilde{\Phi}_i(\tilde{h}) = 0$ ($i=1,2,\dots,m_r$); then $\tilde{h} \in R_A(\widetilde{\mathfrak{B}}_{r'})$ for all $r' \in (r-\eta(r), r)$, and consequently $\tilde{h} \in R_A(\widetilde{\mathfrak{A}}_r)$.

This proves that the operator A from $\mathfrak{A}_r$ into $\widetilde{\mathfrak{A}}_r$ is normally solvable and has finite d-characteristic ((1), p. 50). Moreover, it has been found that

$$\alpha_A(\mathfrak{A}_r) = \lim_{r'\to r-0} \alpha_A(\mathfrak{B}_{r'}) \quad \text{and} \quad \beta_A(\widetilde{\mathfrak{A}}^*_r) = \lim_{r'\to r-0} \beta_A(\widetilde{\mathfrak{B}}^*_{r'}).$$

The index

$$\varkappa_A(\mathfrak{A}_r) = \beta_A(\widetilde{\mathfrak{A}}_r^*) - \alpha_A(\mathfrak{A}_r)$$

is an integer-valued nonincreasing function of $r \in (a, b)$, and

$$\varkappa_A(\mathfrak{A}_r) = \lim_{r' \to r-0} \varkappa_A(\mathfrak{B}_{r'}).$$

Remark. The theorem is also true in the case when A is a Φ-operator from $\mathfrak{B}_r$ into $\widetilde{\mathfrak{B}}_r$ for all $r \in (a, b)$ except for a discrete set of values.

On the basis of this theorem and Theorem 2.2 of (1), we obtain:

Theorem 2. Let A be an operator satisfying the conditions of Theorem 1. Then there is a positive function $\rho(r)$ $(a < r < b)$ such that, whatever the linear bounded operator B mapping $\mathfrak{B}_r$ into $\widetilde{\mathfrak{B}}_r$, independently of $r \in (a, b)$, for which* ps $\|B\|_r < \rho(r)$, the operator $A + B$, acting from $\mathfrak{A}_r$ into $\widetilde{\mathfrak{A}}_r$ $(a < r < b)$, will also be a Φ-operator, and

$$\varkappa_{A+B}(\mathfrak{A}_r) = \varkappa_A(\mathfrak{A}_r) \quad (a < r < b).$$

With the help of analogous arguments and Theorem 2.4 of (1) we obtain:

Theorem 3. Let the operator A satisfy the conditions of Theorem 1. Then there is a positive function $\rho(r)$ $(a < r < b)$ such that for all linear bounded operators B, acting from $\mathfrak{B}_r$ into $\widetilde{\mathfrak{B}}_r$ independently of r, for which $\|B\|_r < \rho(r)$ $(a < r < b)$, the operator $A + B$, acting from $\mathfrak{A}_r$ into $\widetilde{\mathfrak{A}}_r$ $(a < r < b)$, will also be a Φ-operator,

$$\varkappa_{A+B}(\mathfrak{A}_r) = \varkappa_A(\mathfrak{A}_r) \quad (a < r < b),$$

and, moreover,

$$\alpha_{A+B}(\mathfrak{A}_r) \leqslant \alpha_A(\mathfrak{A}_r) \quad (a < r < b).$$

Theorem 4. If A is a $\Phi_\pm$-operator ((1), p. 89) from $\mathfrak{B}_r$ into $\widetilde{\mathfrak{B}}_r$ for $a < r < b$, then it is respectively also a $\Phi_\pm$-operator from $\mathfrak{A}_r$ into $\widetilde{\mathfrak{A}}_r$ for $a < r < b$.

Using Theorems 7.1 and 7.2 of (1), we obtain:

* By ps $\|B\|_r$ we denote the pseudonorm of the operator B from $\mathfrak{B}_r$ into $\widetilde{\mathfrak{B}}_r$ (see (1), p. 55).

Theorem 5. Let A be a Φ_+-operator (Φ_--operator) acting from $\mathfrak{B}_r$ into $\widetilde{\mathfrak{B}}_r$, independently of $r \in (a,b)$. Then there exists a function $\rho(r) > 0$ such that, whatever the bounded linear operator B may be ($B\mathfrak{B}_r \subset \widetilde{\mathfrak{B}}_r$), whose action does not depend on $r \in (a,b)$, if $\|B\|_r < \rho(r)$ ($a < r < b$), the operator $A + B$, acting from $\mathfrak{A}_r$ into $\widetilde{\mathfrak{A}}_r$, will also be a Φ_+-operator (Φ_--operator), and moreover

$$\alpha_{A+B}(\mathfrak{A}_r) \le \alpha_A(\mathfrak{A}_r) \quad \text{for } r \in (a,b)$$

$$(\beta_{A+B}(\widetilde{\mathfrak{A}}_r) \le \beta_A(\widetilde{\mathfrak{A}}_r) \quad \text{for } r \in (a,b)).$$

Remark. The theorems stated will remain valid if A is assumed to be a bounded linear operator mapping $\mathfrak{B}_r$ into $\widetilde{\mathfrak{B}}_r$ ($a < r < b$), and the spaces $\mathfrak{A}_r$ are replaced by the spaces $\overline{\mathfrak{A}}_r$, consisting of elements of $\sum_{r''>r} \mathfrak{B}_{r''}$ with the following definition of convergence: $f_n \to f$ if and only if there exists an $r'' > r$ such that $\{f_n\}_1^\infty$ and f are contained in $\mathfrak{B}_{r''}$ and $\|f_n - f\|_{r''} \to 0$. In this case the conjugate space $\overline{\mathfrak{A}}_r^*$ is equal to

$$\prod_{r''>r} \mathfrak{B}^*_{r''}, \qquad \alpha_A(\overline{\mathfrak{A}}_r) = \lim_{r'' \to r+0} \alpha_A(\mathfrak{B}_{r''})$$

and

$$\beta_A(\widetilde{\mathfrak{A}}^*_r) = \lim_{r'' \to r+0} \beta_A(\widetilde{\mathfrak{B}}^*_{r''}).$$

2. We shall show one possible application of the preceding theorems. Consider the Banach space $\mathfrak{B}_r$ ($0 < r < R$), consisting of all

$$f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

for which

$$\|f\|_r = \sum_n |a_n| r^n < \infty.$$

$\mathfrak{B}_r$ is a normed ring. The space $\mathfrak{B}^*_r$ may be realized as the set of functions

$$\Phi = \Phi(\zeta) = \sum_{n=0}^{\infty} c_n \zeta^{-n-1} \quad (|\zeta| > r)$$

with

$$\|\Phi\|_r = \sup_n \frac{|c_n|}{r^n} < \infty.$$

The action of the functional is given by the formula

$$\Phi(f) = \sum_n a_n c_n = \frac{1}{2\pi i} \int_{|\zeta|=r} \Phi(\zeta) f(\zeta)\, d\zeta$$

(in the case where the integral does not exist, its value is taken to be equal to $\Phi(f)$). $\mathfrak{A}_r$ and $\overline{\mathfrak{A}}_r$ are spaces of analytic functions in $|z| < r$, respectively $|z| \le r$, with their usual topologies.

The space $\mathfrak{B}_r$ possesses in the domain $|z| \le r$ a reproducing family of functionals

$$\{\Phi_z\}_{|z|<r} = \left\{\frac{1}{\zeta - z}\right\}_{|z|<r},$$

so that $\Phi_z(f) = f(z)$ for $f \in \mathfrak{B}_r$ and $|z| \le r$. Moreover, for any $|z| < r$ and natural $n \ge 1$, there exists a linear functional

$$\Phi_z^{(n)} = \frac{n!}{(\zeta - z)^{n+1}}$$

such that

$$\Phi_z^{(n)}(f) = f^{(n)}(z).$$

To every bounded linear operator B, $B\mathfrak{B}_r \subset \mathfrak{B}_r$, there corresponds a kernel

$$B^*\Phi_z = B^*\frac{1}{\zeta - z} = \mathfrak{B}(z,\zeta) = \sum_n \sum_m \varepsilon_{mn} z^m \zeta^{-n-1}$$

such that:

1) $\mathfrak{B}(z,\zeta)$, for $|z| \le r$, belongs to $\mathfrak{B}_r^*$;
2) $\mathfrak{B}(z,\zeta) \in \mathfrak{B}_r \subset \mathfrak{B}_r^{**}$ for $|\zeta| > r$;
3) $$\sup_n \sum_m |\varepsilon_{mn}| r^{m-n} = \|B\|_r;$$
4) for all $f(z) \in \mathfrak{B}_r$ and $\Phi(\zeta) \in \mathfrak{B}_r^*$ we have

$$Bf(z) = \frac{1}{2\pi i}\int_{|\zeta|=r} \mathfrak{B}(z,\zeta) f(\zeta)\, d\zeta, \qquad B^*\Phi(\zeta) = \frac{1}{2\pi i}\int_{|z|=r} \mathfrak{B}(z,\zeta)\Phi(z)\, dz, \tag{1}$$

where the integrals are understood as the actions of functionals represented by the corresponding kernel.

Consider in $\mathfrak{B}_r$, $r \in (0, R)$, the operator A of multiplication by a function $p(z)$, analytic in $|z| < R$. Exclude the discrete set of values r of the interval $(0, R)$ for which $p(re^{i\theta})$ vanishes. For the remaining values

of the interval $(0, R)$

$$p(z) = \prod_{i=1}^{k_r} (z - z_i)^{\nu_i} q_r(z) \qquad \left(|z_i| < r,\ \sum_{i=1}^{k_r} \nu_i = m_r,\ q_r(z) \ne 0,\ \text{for } |z| \le r\right).$$

To the operator A there corresponds the kernel

$$\mathfrak{A}(z,\zeta)=\frac{p(z)}{\zeta-z}$$

and

$$\|A_r\|=\|p(z)\|_r.$$

A is a Φ-operator from $\mathfrak{B}_r$ into $\mathfrak{B}_r$ $(0<r<R,\ p(re^{i\theta})\neq 0)$, with $\alpha_A(\mathfrak{B}_r)=0$ and $\beta_A(\mathfrak{B}_r)=m_r$. The subspace $R_A(\mathfrak{B}_r)$ is characterized by the fact that its elements are orthogonal to the functionals $\Phi_{z_i}^{(s)}$ $(i=1,2,\dots,k_r;\ s=0,1,\dots,\nu_i-1)$. On $R_A(\mathfrak{B}_r)$ the operator has a bounded inverse A^{-1}, with

$$\|A^{-1}\|_r\leq\|q_r^{-1}(z)\|_r\left\{\prod_{i=1}^{k_r}(r-|z_i|)^{\nu_i}\right\}^{-1}.$$

For the operator A theorem 1 holds.

Let us also consider a linear bounded operator B, $B\mathfrak{B}_r\subset\mathfrak{B}_r$, with kernel

$$\mathfrak{B}(z,\zeta)=\sum_n\sum_m\varepsilon_{mn}z^m\zeta^{-n-1},$$

satisfying the condition

$$\overline{\lim_{n\to\infty}}\sum_m|\varepsilon_{mn}|r^{m-n}<\prod_{i=1}^{k_r}(r-|z_i|)^{\nu_i}\,\|q_r^{-1}(z)\|_r^{-1} \tag{2}$$

for all nonexceptional values of r. Then the operators A and B satisfy the conditions of theorem 2. If one assumes

$$\sup_n\sum_m|\varepsilon_{mn}|r^{m-n}<\prod_{i=1}^{k_r}(r-|z_i|)\|q_r^{-1}(z)\|_r^{-1},$$

then A and B satisfy the conditions of theorem 3.

From the preceding, if we put $\varepsilon_{mn}=0$ for $m\leq n$, $p(0)\neq 0$, and replace condition (2) by the condition

$$\lim_{n\to\infty}\sum_m\varepsilon_{mn}r^{m-n}=0,$$

we obtain the results of M. A. Evgrafov (see ([2]), Ch. IV, §§ 2, 3).

The scheme used above may also be applied to the space of analytic functions of many variables $f(z_1, z_2, \dots, z_k)$ $(|z_i| < R_i)$ by introducing the spaces $\mathfrak{B}_{r_1,\dots,r_k}$ $(r_i < R_i)$ of all functions

$$f = \sum_{n_i} a_{n_1,\dots,n_k} z_1^{n_1} \cdots z_k^{n_k},$$

for which

$$\|f(z_1, \dots, z_k)\|_{r_1,\dots,r_k} = \sum_{n_i} |a_{n_1,\dots,n_k}| r_1^{n_1} \cdots r_k^{n_k} < \infty.$$

The operator A should be replaced by the operator of multiplication by the function

$$p_1(z_1) \cdots p_k(z_k) q(z_1, \dots, z_k),$$

where the $p_i(z_i)$ are analytic for $|z_i| < R_i$ $(i = 1, 2, \dots, k)$, and $q(z_1, \dots, z_k)$ is a nonvanishing analytic function in the polycylinder $|z_i| < R_i$ $(i = 1, 2, \dots, k)$. The operator A will no longer be a Φ-operator, but will remain a Φ_+-operator.

Let us also note that, with the aid of the preceding theorems, one can formulate certain criteria of completeness and basis property for systems of functions of the form $\{(A + B) f_n(z)\}_0^\infty$ in an analytic space.

Chernivtsi State University

Received
26 IV 1958.

References

[1] I. Ts. Gokhberg, M. G. Krein, *Uspekhi Mat. Nauk*, **12**, no. 2 (1957). [2] M. A. Evgrafov, *Trudy Moskov. Mat. Obshch.*, **5**, 89 (1956).

Note: Figure translations are in progress. See original paper for figures.