MATHEMATICS

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Abstract

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ON THE GENERAL THEORY OF EQUATIONS WITH DEVIATING ARGUMENT

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§ 1. **Statement of the problem and classification.** Differential equations with deviating argument are differential equations in which the unknown function and its derivatives enter at different values of the argument:

$$\Phi(t, y(\alpha_0(t)), \dots, y^{(m_0)}(\alpha_0(t)), y(\alpha_1(t)), \dots, y^{(m_1)}(\alpha_1(t)), \dots$$
$$\dots, y(\alpha_n(t)), \dots, y^{(m_n)}(\alpha_n(t))) = 0. \quad (1)$$

The functions $\alpha_0(t), \alpha_1(t), \dots, \alpha_n(t)$ are given. By the derivative $y^{(j)}(\alpha_i(t))$ is meant the j-th derivative of the function $y(z)$, taken at the value of the argument $z = \alpha_i(t)$.

Let $\alpha(t) = \max_{0 \le i \le n}[\alpha_i(t)]$. We shall assume that $\alpha_i(t)$ $(i = 0, 1, 2, \dots, n)$ are continuous and such that the range of variation of t can be divided into segments $[t_\nu, t_{\nu+1}]$ on which $\alpha(t)$ is equal to one of the functions $\alpha_i(t)$ and is monotone.

Consider a fixed segment $[t_\nu, t_{\nu+1}]$. Without loss of generality one may assume that $\alpha(t) = \alpha_0(t)$ for $t \in [t_\nu, t_{\nu+1}]$. Introduce the new independent variable $x = \alpha(t)$. Since $\alpha_0(t) \ge \alpha_i(t)$ $(i = 1, 2, \dots, n)$ for $t \in [t_\nu, t_{\nu+1}]$, the remaining functions $\alpha_i(t)$ can be represented in the form $\alpha_i(t) = x - \Delta_i(x)$, where $\Delta_i(x) \ge 0$.

Equation (1) is thereby reduced to the form

$$F(x, y(x), \dots, y^{(m_0)}(x), y(x - \Delta_1(x)), \dots, y^{(m_1)}(x - \Delta_1(x)), \dots$$
$$\dots, y(x - \Delta_n(x)), \dots, y^{(m_n)}(x - \Delta_n(x))) = 0. \quad (2)$$

Denote $\mu = \max_{1 \le i \le n}[m_i]$. Fix a point A_0, called the initial point. Each function $\Delta_i(x)$ defines an initial set E_0^i, consisting of the point A_0 and those points $x - \Delta_i(x)$ for which $x \ge A_0$ and $x - \Delta_i(x) < A_0$.

Put $E_0 = \bigcup_{i=1}^{n} E_0^i$. On E_0 we prescribe an initial function $\varphi(x)$ differentiable μ times. Denote $y_i = \varphi^{(i)}(A_0)$ $(i = 0, 1, 2, \dots, \mu)$. If $\mu < m_0 - 1$, then we prescribe additionally the numbers $y_{\mu+1}, y_{\mu+2}, \dots, y_{m_0-1}$. If A_0 is isolated in E_0, then $y_0, y_1, \dots, y_{m_0-1}$ are prescribed arbitrarily.

A function $y(x)$, defined for $A_0 \leq x < B$, is called a solution of equation (2) with initial data $\varphi(x), y_0, y_1, \dots, y_{m_0-1}$, if* $y(A_0) = y_0, y'(A_0) = y_1, \dots, y^{(m_0-1)}(A_0) = y_{m_0-1}$ and $y(x)$ satisfies–

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* By a derivative at the end of an interval here and everywhere below we shall mean a one-sided derivative.

satisfies equation (2) for $A_0 \leqslant x < B$, and, if $x \geqslant A_0$ while $x - \Delta_i(x) < A_0$, then in equation (2) one should substitute $\varphi_i^{(j)}(x - \Delta_i(x))$ in place of $y^{(j)}(x - \Delta_i(x))$. (One may prescribe on each E_0^i $m_i + 1$ arbitrary functions $\varphi_{i0}(x)$, $\varphi_{i1}(x), \dots, \varphi_{im_i}(x)$, and, when $x - \Delta_i(x) < A_0$, put $y^{(j)}(x - \Delta_i(x)) = \varphi_{ij}(x - \Delta_i(x))$. The numbers $y_0, y_1, \dots, y_{m_0-1}$ may also be prescribed arbitrarily.)

Here we consider the case when equation (2) is solvable with respect to $y^{(m_0)}(x)$:

$$y^{(m_0)}x = f(x, y(x), \dots, y^{(m_0-1)}(x), y(x - \Delta_1(x)), \dots$$
$$\dots, y^{(m_1)}(x - \Delta_1(x)), \dots, y(x - \Delta_n(x)), \dots, y^{(m_n)}(x - \Delta_n(x))). \qquad (3)$$

Denote $\lambda = m_0 - \mu$.

Equations for which $\lambda > 0$ are called **equations with retarded argument**. Equations for which $\lambda = 0$ are called **equations of neutral type**. Equations for which $\lambda < 0$ we shall call **equations of advanced type**. In particular, equations with advanced argument (see (1), Ch. 5, § 1), in the case where they can be reduced to the form (3), turn out to be equations of advanced type.

§ 2. **Existence, uniqueness, and smoothness of solutions and continuous dependence on the initial data in the case of applicability of the method of successive integration.** The method of successive integration (the step method), for the case when the right-hand side of the equation depends on a single delay function, is known (see, for example, (1), Ch. 5, § 2). In this paragraph the method of successive integration is applied in the case when the right-hand side of the equation depends on several delay functions.

The functions $\Delta_1(x), \Delta_2(x), \dots, \Delta_n(x)$ are continuous.

Let (A_0, A_1) be the maximal interval with left endpoint at the point A_0 such that, if $x \in (A_0, A_1)$, then $x - \Delta_i(x) \in E_0$ $(i = 1, 2, \dots, n)$. Denote

$$E_1 = (A_0, A_1], \qquad F_1 = E_0 \cup E_1.$$

Suppose that in this way $A_1, A_2, \dots, A_{k-1}$ and the sets $E_1, E_2, \dots, E_{k-1}$ and $F_1, F_2, \dots, F_{k-1}$ have been defined. Then (A_{k-1}, A_k) is the maximal interval with left endpoint at the point A_{k-1} such that, if $x \in (A_{k-1}, A_k)$, then $x - \Delta_i(x) \in F_{k-1}$ $(i = 1, 2, \dots, n)$,

$$E_k = (A_{k-1}, A_k], \qquad F_k = \bigcup_{i=0}^{k} E_i.$$

In order that the segment $[A_0, B]$ be covered by a finite number of sets E_k, it is necessary and sufficient that, for each $i = 1, 2, \dots, n$, either $\Delta_i(A_0) > 0$, or $\Delta_i(A_0) = 0$, but there exist a segment $[A_0, A_1^i]$ such that $x - \Delta_i(x) < A_0$ for $x \in [A_0, A_1^i]$, and that $\Delta_i(x) > 0$ $(i = 1, 2, \dots, n)$ for $x \in (A_0, B]$.

The solution is found successively on the sets $E_1, E_2, \dots, E_k, \dots$; moreover, at each step, finding the solution of equation (3) reduces to integrating an ordinary differential equation without delays. If at each step the solution of the Cauchy problem is unique, then the solution of the Cauchy problem for equation (3) is unique.

Denote by X_1 the set of all such points x that $x - \Delta_i(x) = A_0$ for at least one of the functions $\Delta_i(x)$; denote by X_2 the set of all such points x that $x - \Delta_i(x)$, for at least one of the functions $\Delta_i(x)$, belongs to X_1; if $X_1, X_2, \dots, X_{j-1}$ are thus defined, then X_j is the set of all such points x that $x - \Delta_i(x)$ belongs to X_{j-1} for at least one of the functions $\Delta_i(x)$. If the functions $\Delta_i(x)$ $(i = 1, 2, \dots, n)$ are different from zero on some segment with left endpoint at the point A_0, and the graph of each of them inter-

intersects any straight line with slope equal to one a finite number of times, then on this segment there lies a finite number of points belonging to the sets X_j.

Put

$$\gamma_i(z) = \sup_t E\{t,\ t - \Delta_i(t) < z,\ t > z\}$$

and $\gamma_i(z) = z$, if this set is empty. Denote

$$\Gamma(z) = \max_{1 \le i \le n} \gamma_i(z).$$

$\Gamma(A_0) = \Gamma_1,\ \Gamma(\Gamma_1) = \Gamma_2, \dots, \dots,\ \Gamma(\Gamma_{k-1}) = \Gamma_k.$

$$H_1 = (A_0, \Gamma_1], \quad H_2 = (\Gamma_1, \Gamma_2], \dots, \quad H_k = (\Gamma_{k-1}, \Gamma_k].$$

By $\widetilde{E}_k$ and $\widetilde{H}_k$ we shall denote the intersections of the sets E_k and H_k with all the sets X_j.

Theorem 1. Let on the set E_0 the initial function be differentiable $l \ge \mu$ times, let the functions f and $\Delta_i(x)$ $(i = 1, 2, \dots, n)$ be differentiable with respect to their arguments $l + (k-1)\lambda - \mu$ times, and let there exist solutions of the ordinary differential equations obtained by applying the method of successive integration. Then, if $\lambda \le 0$, the solution $y(x)$ of equation (3) is differentiable $l + k\lambda$ times on the set $\widetilde{E}_k$; if $\lambda \geq 0$, then the solution $y(x)$ of equation (3) is differentiable $l + k\lambda$ times on the set $\widetilde{H}_k$.

Theorem 2. Let $\varphi(x), y_0, y_1, \dots, y_{m_0-1}$ and $\overline{\varphi}(x), \overline{y}_0, \overline{y}_1, \dots, \overline{y}_{m_0-1}$ be initial data, and let $y(x)$ and $\overline{y}(x)$ be the corresponding solutions, and suppose that the conditions of Theorem 1 are fulfilled and, in addition, the solutions of the ordinary differential equations obtained by applying the method of successive integration depend continuously on the initial conditions. Then, whatever $\varepsilon > 0$ and the natural number k may be, there exists a number $\delta = \delta(\varepsilon, k) > 0$ such that, if

$$|\varphi^{(j)}(x) - \overline{\varphi}^{(j)}(x)| < \delta \quad (j = 0, 1, \dots, l;\ l \geq \mu), \qquad |y_j - \overline{y}_j| < \delta$$

$$(j = 0, 1, \dots, m_0 - 1),$$

then

$$|y^{(j)}(x) - \overline{y}^{(j)}(x)| < \varepsilon \quad (j = 0, 1, \dots, l + k\lambda)$$

for $x \in \widetilde{E}_k$ when $\lambda \leq 0$, and for $x \in \widetilde{H}_k$ when $\lambda \geq 0$.

§ 3. Existence, uniqueness, and continuous dependence on the initial data of solutions of differential equations of neutral type in the case when the method of successive integration is not applicable*

We shall consider a system of n first-order equations, which in vector form is written as the equation

$$\begin{gathered} y'(x) = G(x, y(x), \dots, y(x - \Delta_i(x)), \dots, y'(x - \Delta_i(x)), \dots \\ \dots, y(x - \Delta_j(x)), \dots, y'(x - \Delta_j(x)), \dots), \end{gathered} \tag{4}$$

where G is a vector function of x and of $2m + 2p + 1$ vectors

$$y(x),\ y(x - \Delta_i(x)),\ y'(x - \Delta_i(x)) \quad (i = 1, 2, \dots, m); \quad y(x - \Delta_j(x)),\ y'(x - \Delta_j(x)) \quad (j = 1, 2, \dots, p).$$

We shall suppose that the delay functions belong to one of two groups. The functions of the first group $\Delta_i(x)$ $(i = 1, 2, \dots, m)$ have the property that $x - \Delta_i(x) > A_0$ on some interval (A_0, B_i) to the right of the point A_0. The functions of the second group $\Delta_j(x)$ $(j = 1, 2, \dots, p)$ are such that either $\Delta_j(A_0) > 0$, and then there necessarily exist intervals (A_0, B_j) to the right of the point A_0 such that $x - \Delta_j(x) < A_0$ for $x \in (A_0, B_j)$, or $\Delta_j(A_0) = 0$, but nevertheless there exist intervals (A_0, B_j) to the right of the point A_0 such that $x - \Delta_j(x) < A_0$ for $x \in (A_0, B_j)$. On the initial set an initial vector function $\varphi(x)$ is prescribed. Denote

$$B = \min_{ij}(B_i, B_j).$$

Let $x \in (A_0, B)$. Then $x - \Delta_j(x) < A_0$, and in place of $y(x - \Delta_j(x))$, $y'(x - \Delta_j(x))$ in equation (4) one may substitute the known

$$\varphi(x - \Delta_j(x)), \quad \varphi'(x - \Delta_j(x)),$$

as a result of which equation (4) can be written in the form

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* For equations with retarded argument the corresponding theorems in this case are proved by the usual methods (see, for example, (1), Chap. 5, § 3).

$$y'(x) = g(x, y(x), \dots, y(x - \Delta_i(x)), \dots, y'(x - \Delta_i(x)), \dots) \tag{5}$$

$$(i = 1, 2, \dots, m).$$

In order that equation (5) have a solution with initial value y_A and with derivative of the solution at the initial point equal to y'_A, it is necessary that y'_A be a real root of the finite vector equation with respect to z

$$z = g(A, y_A, y'_A, \dots, y_A, z, \dots, z). \tag{6}$$

Denote

$$\mu^i_\xi = \inf_{A_0 \le x \le \xi} D\Delta_i(x),$$

where $D\Delta_i(x)$ is the lower derivative number of the function $\Delta_i(x)$ at the point x.

If z is an n-dimensional vector, then $|z| = \max_{1 \le i \le n} |z_i|$.

Theorem 3. *Let y'_A be a real root of equation (6), and let the function $g(u_1, u_2, \dots, u_{2m+2})$ satisfy a Lipschitz condition in $u_1, u_2, \dots, u_{2m+2}$, with constants $p, q, r_1, r_2, \dots, r_m, s_1, s_2, \dots, s_m$, in some neighborhood of the point $(A, y_A, y_A, \dots, y_A, y'_A, \dots, y'_A)$, and suppose there exists $\xi \in (A, B]$ such that $\mu^i_\xi > 0$ $(i = 1, 2, \dots, m)$ and*

$$\lambda = \max_{1 \le i \le m} s_i(1 - \mu^i_\xi) < \frac{1}{m}. \tag{7}$$

Then, on some half-interval $[A_0, A_0 + h)$, there exists a unique continuously differentiable solution $y(x)$ of equation (5) with initial value y_A, with derivative equal to y'_A at the point A_0, and satisfying a Lipschitz condition on $[A_0, A_0 + h)$.

Theorem 3 is a generalization of the theorem proved in paper [2], which is obtained from it when $m = 1$. That paper also gives an example in which, already for $s(1 - \mu_\xi) = 1$, uniqueness is violated.

Suppose now that in equation (5) $m = 2$ and the delay functions are related by

$$\Delta_2(x) = \Delta_1(x) + \Delta_1(x - \Delta_1(x)).$$

Then in Theorem 3 condition (7) may be replaced by the condition $s(1 - \mu_\xi) < \frac{1}{2}(\sqrt{5} - 1)$, where $s = \max[s_1, s_2, 1]$. Equations of this kind lead to the Euler equations in solving variational problems in the case when the integrand depends on a single delay (see [1], Ch. 6, § 3).

Theorem 4. *Suppose the conditions of Theorem 3 are satisfied and $\bar{y}'_A$ is a root of equation (6) corresponding to the value $\bar{y}_A$. Suppose equation (6) defines z as a continuous function of y_A in some neighborhood of $\bar{y}_A$. If this function is multivalued, then suppose that the branch on which $\bar{y}'_A$ lies is continuous.*

Then, if an arbitrary number $\varepsilon > 0$ and some X, $A_0 < X \le A_0 + h$, are given, there exists $\delta > 0$ such that, if $|y_A - \bar{y}_A| < \delta$, then there exists a root y'_A of equation (6), corresponding to the value y_A, such that for the solutions $y(x)$ and $\bar{y}(x)$ of equation (5) with initial values y_A and $\bar{y}_A$ and derivatives at the initial point equal to y'_A and $\bar{y}'_A$, one has $|y(x) - \bar{y}(x)| < \varepsilon$ and $|y'(x) - \bar{y}'(x)| < \varepsilon$ for $A_0 \le x \le X$.

If, after applying Theorem 3 or 4, the point $A_0 + h$ is taken as the initial point, then the method of successive integration becomes possible and one may apply Theorem 1 or 2.

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References

1. L. E. Elsgolts, *Qualitative Methods in Mathematical Analysis*, 1955.

2. G. A. Kamenskii, *Uch. zap. MGU*, Math., 8, issue 181, 83 (1956).

Note: Figure translations are in progress. See original paper for figures.