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MATHEMATICS

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Abstract

Full Text

MATHEMATICS

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ON MULTIPLE DIFFERENTIATION WITH RESPECT TO A PARAMETER OF SOLUTIONS OF SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS WITH A SMALL PARAMETER AT THE DERIVATIVE

(Presented by Academician I. G. Petrovskii, 9 X 1957)

In papers ⁽¹⁻⁵⁾ a system of differential equations was considered

$$\begin{aligned} \mu \frac{dz}{dt} &= F(z, x, t), \\ \frac{dx}{dt} &= f(z, x, t), \\ z|_{t=0} &= z^0, \quad x|_{t=0} = x^0, \end{aligned} \tag{1}$$

where $\mu > 0$ is a small parameter. In ^(1,2) it was shown that, as $\mu \rightarrow 0$, the solution $z(t, \mu)$, $x(t, \mu)$ of system (1) tends to the solution $\bar{z}(t)$, $\bar{x}(t)$ of the degenerate system

$$\begin{aligned} F(\bar{z}, \bar{x}, t) &= 0, \\ \frac{d\bar{x}}{dt} &= f(\bar{z}, \bar{x}, t), \\ \bar{x}|_{t=0} &= x^0. \end{aligned} \tag{2}$$

In the case when the equation $F(z, x, t) = 0$ has several roots $z = \varphi(x, t)$, in constructing the solution z of the degenerate system (2), a single root is selected in the following way.

We shall call a root $z = \varphi(x, t)$ of the equation $F(z, x, t) = 0$ **stable** in a domain D of the space (x, t) if

$$\frac{\partial F}{\partial z}(\varphi(x, t), x, t) < 0$$

for all points (x, t) from \bar{D} . We shall call the **domain of influence of a stable root** $z = \varphi(x, t)$ the set of points (z^0, x^0, t^0) for which

$$\text{sign } F(z, x^0, t^0) = \text{sign } F(z^0, x^0, t^0)$$

for

$$z^0 \leq z \leq \varphi(x^0, t^0),$$

i.e.

$$F(z, x^0, t^0) > 0$$

for $z^0 < \varphi(x^0, t^0)$, and

$$F(z, x^0, t^0) < 0$$

for $z^0 > \varphi(x^0, t^0)$.

In constructing the solution z of the degenerate system (2), one takes that stable root whose domain of influence contains the initial point $z^0, x^0, 0$.

In papers (3–5) the differentiability with respect to μ of the solution $z(t, \mu), x(t, \mu)$ of system (1) was studied under the condition of the existence of continuous first-order partial derivatives for $f(z, x, t)$ and second-order partial derivatives for $F(z, x, t)$ with respect to all arguments. The derivatives z_μ, x_μ satisfy the system of equations

$$\begin{aligned} \mu \frac{d}{dt} z_\mu + \frac{d}{dt} z &= F_z(z(t, \mu), x(t, \mu), t) z_\mu + F_x(z(t, \mu), x(t, \mu), t) x_\mu, \\ \frac{d}{dt} x_\mu &= f_z(z(t, \mu), x(t, \mu), t) z_\mu + f_x(z(t, \mu), x(t, \mu), t) x_\mu, \\ z_\mu|_{t=0} &= 0, \quad x_\mu|_{t=0} = 0. \end{aligned} \quad (3)$$

The corresponding degenerate system has the form

$$\begin{aligned} \frac{d}{dt} \bar{z} &= F_z(\bar{z}, \bar{x}, t) \bar{z}_\mu + F_x(\bar{z}, \bar{x}, t) \bar{x}_\mu, \\ \frac{d}{dt} \bar{x}_\mu &= f_z(\bar{z}, \bar{x}, t) \bar{z}_\mu + f_x(\bar{z}, \bar{x}, t) \bar{x}_\mu. \end{aligned} \quad (4)$$

In (3–5) it is shown that $\lim_{\mu \rightarrow 0} z_\mu = \bar{z}_\mu$, $\lim_{\mu \rightarrow 0} x_\mu = \bar{x}_\mu$, if by \bar{z}_μ, \bar{x}_μ one means the solution of (4) that satisfies the special initial condition

$$\bar{x}_\mu|_{t=0} = \bar{x}_\mu^0 = \int_0^\infty \{f(z_0(\tau), x^0, 0) - f(\varphi(x^0, 0), x^0, 0)\} d\tau, \quad (5)$$

where $z_0(\tau)$ is the solution of the equation $dz_0/d\tau = (z_0, x^0, 0)$, satisfying the initial condition $z_0|_{t=0} = z^0$. Let us note that $x_\mu|_{t=0} = 0$, while $\bar{x}_\mu^0 = \bar{x}_\mu|_{t=0}$, generally speaking, is not equal to zero, in contrast to the property of the solution itself, which consists in the fact that the initial values x and \bar{x} coincide, $x|_{t=0} = \bar{x}|_{t=0} = x^0$. The quantity \bar{x}_μ^0 , expressed by the integral (5), may be called the jump of the initial value for the first derivative x_μ .

In the present note a result will be formulated concerning the investigation of the derivatives of order n of the solution of system (1) with respect to the parameter μ . We shall assume that $F(z, x, t)$ has continuous partial derivatives with respect to all arguments up to order $(n+1)$ inclusive, and $f(z, x, t)$ has continuous partial derivatives with respect to all arguments up to order n inclusive.

The derivatives of order n of the solution of system (1), $z_{\mu n}, x_{\mu n}$, satisfy the system

$$\mu \frac{d}{dt} z_{\mu n} + n \frac{d}{dt} z_{\mu, n-1} = F^{(n)}(z, x, t, z_\mu, \dots, z_{\mu n}, x_\mu, \dots, x_{\mu n}), \quad (6a)$$

$$\frac{d}{dt} x_{\mu n} = f^{(n)}(z, x, t, z_\mu, \dots, z_{\mu n}, x_\mu, \dots, x_{\mu n}), \quad z_{\mu n}|_{t=0} = 0, \quad x_{\mu n}|_{t=0} = 0, \quad (6b)$$

where the right-hand sides are certain functions, linear with respect to $z_{\mu n}, x_{\mu n}$, with coefficients depending on $z(t, \mu), x(t, \mu), t$.

It is essential that the coefficient of $z_{\mu n}$ in equation (6a) is the quantity $F_z(z(t, \mu), x(t, \mu), t) < 0$ for sufficiently small μ .

If one assumes that the limiting functions for the derivatives of lower order are known, then one can construct the degenerate system corresponding to (6) (we denote the limiting values of the derivatives by the same symbol with a bar above):

$$n \frac{d}{dt} \bar{z}_{\mu, n-1} = F^{(n)}(\bar{z}, \bar{x}, t, \bar{z}_\mu, \dots, \bar{z}_{\mu n}, \bar{x}_\mu, \dots, \bar{x}_{\mu n}),$$

$$\frac{d}{dt} \bar{x}_{\mu n} = f^{(n)}(\bar{z}, \bar{x}, t, \bar{z}_\mu, \dots, \bar{z}_{\mu n}, \bar{x}_\mu, \dots, \bar{x}_{\mu n}). \quad (7)$$

Making in (1) the change of variables $t/\mu = \tau$, we obtain

$$\frac{dz}{d\tau} = F(z, x, \tau\mu),$$

$$\frac{dx}{d\tau} = \mu f(z, x, \tau\mu),$$

$$z|_{\tau=0} = z^0, \quad x|_{\tau=0} = x^0.$$

By virtue of the smoothness conditions imposed above on the right-hand sides of (1), and the stability condition, one can, on the basis of the known theorems on the dependence of solutions of systems of differential equations on a parameter, determine, in the right-hand side, from the corresponding variational equations, at least n terms of the expansion of the solution in a power series in μ

$$z \sim z_0(\tau) + \mu z_1(\tau) + \dots + \mu^n z_n(\tau),$$

$$x \sim x_0(\tau) + \mu x_1(\tau) + \dots + \mu^n x_n(\tau),$$

knowing which, one can also write n terms of the expansion of the function $f(z, x, \tau\mu)$:

$$f \sim f_0(\tau) + \mu f_1(\tau) + \dots + \mu^n f_n(\tau).$$

Theorem. The limiting values $\bar{z}_{\mu^n}, \bar{x}_{\mu^n}$ of the derivatives of order n with respect to the parameter μ of the solution $z(t, \mu), x(t, \mu)$ of system (1) exist on the interval $0 < t \leq T$, where T does not depend on μ , and satisfy the degenerate system of equations (7) with the initial condition

$$\bar{x}_{\mu^n}|_{t=0} = \bar{x}_{\mu^n}^0 = \int_0^\infty (-1)^n \tau^n f_{n-1}^{(n)}(\tau) d\tau. \quad (8)$$

Since the theorem on limiting values for the first derivatives z_μ, x_μ was proved earlier (3-5), the theorem formulated here completely determines the limiting behavior of derivatives of arbitrary order, with respect to the parameter μ , of the solution of system (1).

It is obvious that for $n = 1$ formula (8) for the jumps of the initial values $x_{\mu.n}$ coincides with (5). Indeed,

$$\bar{x}_\mu^0 = - \int_0^\infty \tau f_0'(\tau) d\tau = -\tau[f_0(\tau) - f_0(\infty)]|_0^\infty + \int_0^\infty [f_0(\tau) - f_0(\infty)] d\tau =$$

$$= \int_0^{\infty} [f(z_0(\tau), x^0, 0) - f(\varphi(x^0, 0), x^0, 0)] d\tau.$$

In conclusion, let us note that the theorem carries over directly to the case of the system

$$\mu \frac{d}{dt} z_j = F_j(z_l, x_k, t), \quad j, l = 1, 2, \dots, n;$$

$$\frac{d}{dt} x_i = f_i(z_l, x_k, t), \quad i, k = 1, 2, \dots, m;$$

$$z_j|_{t=0} = z_j^0 \quad x_i|_{t=0} = x_i^0$$

under the same smoothness conditions for the functions F_j, f_i as were formulated above for F, f , and under the stability condition used in (5).

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CITED LITERATURE

- ¹ A. N. Tikhonov, *Matem. sborn.*, **22** (64), No. 2 (1948).
- ² A. N. Tikhonov, *Matem. sborn.*, **31** (73), No. 3 (1952).
- ³ A. B. Vasil'eva, *Matem. sborn.*, **28** (70), No. 1 (1951).
- ⁴ A. B. Vasil'eva, *Matem. sborn.*, **31** (73), No. 3 (1952).
- ⁵ A. B. Vasil'eva, *Vestn. MGU*, No. 3 (1954).

Note: Figure translations are in progress. See original paper for figures.

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