ON DETERMINING THE INTENSITY OF HEAD WAVES IN THE THEORY OF ELASTICITY BY THE RAY METHOD

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Abstract

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ON DETERMINING THE INTENSITY OF HEAD WAVES IN THE THEORY OF ELASTICITY BY THE RAY METHOD

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The problem is considered, in the ray approximation ([1]), of determining the intensity and form of head waves arising at a plane interface between elastic media, in the case where a linearly polarized wave with a front of arbitrary shape is incident on this interface. For acoustics, the question of determining the intensity of head waves in the case of incidence of a spherical wave on a plane interface was considered by Friedrichs and Keller ([2]).

Let two half-spaces $z > 0$ and $z < 0$ be given in rectangular coordinates x, y, z. The half-space $z > 0$ is filled with an elastic medium with velocities of propagation of longitudinal and transverse waves respectively v_{p1} and v_{s1}, and density ρ_1. The elastic characteristics of the medium filling the half-space $z < 0$ will be denoted by v_{p2}, v_{s2}, and ρ_2. At the interface $z = 0$, the conditions of continuity of displacements and stresses are assumed to be satisfied.

We shall represent the displacement vectors of waves propagating in our system in the form of expansions

$$\mathbf{U}^{(\nu)}(x, y, z, t) = \operatorname{Im} \sum_{n=0}^{\infty} \mathbf{U}_n^{(\nu)}(x, y, z) f_n(t - \tau_\nu), \tag{1}$$

where the functions

$$f_n(t) = \int_{\omega_0}^{\infty} F(\omega) \frac{e^{i\omega t}}{[i\omega]^{n+1}} \, d\omega \quad (\omega_0 > 0),$$

which satisfy the recurrence relations $f'_{n+1}(t) = f_n(t)$, characterize the type of discontinuity of the vectors $\mathbf{U}^{(\nu)}(x, y, z, t)$ on the corresponding fronts $t = \tau_\nu(x, y, z)$. The quantities $\tau_\nu(x, y, z)$ are the eikonals of the waves, and the vectors $\mathbf{U}_n^{(\nu)}(x, y, z)$ are the complex measures of discontinuities of different order. The first nonzero vectors $\mathbf{U}_k^{(\nu)}(x, y, z) = U_k^{(\nu)}(x, y, z)\mathbf{t}_k^{(\nu)}$ in the sums (1) have the polarization usual for the type of the given wave (normal to the front for longitudinal waves and tangential for transverse waves). The variation of the lengths of these vectors along rays is determined, as is known ([1]), by the geometrical divergence of the "ray tubes" and, for a homogeneous medium, may be expressed by the formula

$$U_k^{(\nu)}(M) = \sqrt{\frac{d\sigma_0}{d\sigma}}\, U_k^{(\nu)}(M_0), \tag{2}$$

where M and M_0 are two points of the ray, and $d\sigma$ and $d\sigma_0$ are the surface elements of the wave front corresponding to these points ([2], [3]).

An expansion of type (1) is the nonstationary analogue of classical expansions in inverse frequencies. The function $f_0(t)$ may be either discon—

both jump discontinuities and gap discontinuities (and even by a functional), and formula (1) represents a generalized solution (in the sense of the theory of generalized functions (4)) of the dynamic equations of the theory of elasticity.

Let $\mathbf{U}^{(0)}(x, y, z, t)$ be the displacement vector of the incident wave, for which the quantities $\mathbf{U}_n^{(0)}(x, y, z)$, $\tau_0(x, y, z)$, and $F(\omega)$ are given, and let $\mathbf{U}^{(\nu)}(x, y, z, t)$ $(\nu = 1, 2, ...)$ be the displacement vectors of the waves arising in the process of reflection–refraction, for which the quantities $\mathbf{U}_n^{(\nu)}(x, y, z)$ and $\tau_\nu(x, y, z)$ are to be determined.

The superposition of the fields of the incident wave and of the waves newly arising at the boundary must satisfy the conditions of continuity of displacements and stresses at $z = 0$. Among the waves $\mathbf{U}^{(\nu)}(x, y, z, t)$ $(\nu = 1, 2, ...)$, besides the reflected and refracted ones, there are head waves and the so-called "surface" waves with a form determined by the functions

$$\tilde{f}_n(t - \tau_\nu) = \int_{\omega_0}^{\infty} F(\omega) e^{-\varkappa\omega|z|} \frac{e^{i\omega(t-\tau_0)}}{[i\omega]^{n+1}}\, d\omega,$$

where the constant $\varkappa > 0$. The amplitudes $\mathbf{U}_n^{(\nu)}$ of the "surface" waves and the eikonal τ_ν do not depend on the coordinate z, while a discontinuity of the displacement vector of these waves occurs only at $z = 0$.

Using the boundary conditions, it is not difficult to determine the amplitudes $\mathbf{U}_n^{(\nu)}$ of all newly arising waves on the plane $z = 0$. To determine the fields of these waves at an arbitrary point of the medium, it is only necessary to continue the quantities $\mathbf{U}_n^{(\nu)}(x, y, z)$ along the rays*.

In the present note we restrict ourselves to determining the first nonzero terms $\mathbf{U}_k^{(\nu)}$ in the expansions (1), whose polarization is known in advance, and the law of continuation along the rays is determined by formula (2).

Fig. 1

Fig. 1

Figure 1: Fig. 1

Introduce at the point O' of intersection of the ray of the incident wave with the plane $z = 0$ a local coordinate system x', y', z'. Direct the axis $O'z'$ parallel to the axis Oz, place the axis $O'x'$ in the plane of incidence of the ray, and take the axis $O'y'$ normal to it. Decompose the displacement vector of the incident wave into two vectors, one of which lies in the plane of incidence, while the other is parallel to the axis $O'y'$.

By virtue of the linearity of the boundary conditions, we may consider these two components independently of one another; moreover, the case of incidence of waves with a displacement vector directed along the axis $O'y'$ is in principle no different from the acoustic case and may be treated as in (2).

For the case considered by us of polarization of the wave in the plane of incidence, the boundary conditions at $z = 0$ have the form

$$\mathbf{U}^+ = \mathbf{U}^-; \qquad \sigma_{z'}^{(1)}(\mathbf{U}^+) = \sigma_{z'}^{(2)}(\mathbf{U}^-); \qquad \tau_{x'z'}^{(1)}(\mathbf{U}^+) = \tau_{x'z'}^{(2)}(\mathbf{U}^-),$$

where σ_z and τ_{xz} are stresses; $\mathbf{U}^+$ and $\mathbf{U}^-$ are the total displacement vectors above and below the boundary, respectively.

Let $v_{p2} > v_{p1} > v_{s2} > v_{s1}$ and let the incident wave be longitudinal. In this case three head waves** arise, one of which is longitudinal, of type $\mathrm{P_1P_2P_1}$ ($\nu = 1$), and two are transverse, of type $\mathrm{P_1P_2S_1}$ ($\nu = 2$).

* The eikonals $\tau_\nu(x, y, z)$ are readily determined by the methods of geometrical seismics; we assume them to be known.

** For a classification of head waves see (6).

and $P_1P_2S_2$ ($\nu = 3$). We shall denote the refracted longitudinal wave by the symbol P_1P_2 and assign it the index $\nu = 4$.

The boundary conditions for the discontinuity coefficients $U_n^{(\nu)}$ have a local character. If they are written at the corresponding point A (see Fig. 1) and the expansions (1) for the vectors $\mathbf{U}^{(\nu)}(x, y, z, t)$ of the waves participating in the boundary conditions are substituted into them, then, equating to zero the sums of the coefficients of the functions $f_0'(t - \tau_\nu)$ and $f_0(t - \tau_\nu)$*, we obtain systems of algebraic equations for $U_k^{(\nu)}$ at $z = 0$:

$$U_k^{(1)} \sin\theta_k^{(1)} + U_k^{(2)} \cos\theta_k^{(2)} - U_k^{(3)} \cos\theta_k^{(3)} - U_k^{(4)} \sin\theta_k^{(4)} = 0,$$

$$U_k^{(1)} \cos\theta_k^{(1)} - U_k^{(2)} \sin\theta_k^{(2)} - U_k^{(3)} \sin\theta_k^{(3)} - U_k^{(4)} \cos\theta_k^{(4)} = 0,$$

$$U_k^{(1)}\frac{\lambda_1+2\mu_1\cos^2\theta_k^{(1)}}{v_{p2}\sin\theta_k^{(1)}}-U_k^{(2)}\frac{2\mu_1\cos\theta_k^{(2)}}{v_{p2}}+U_k^{(3)}\frac{2\mu_2\cos\theta_k^{(3)}}{v_{p2}}-U_k^{(4)}\frac{\lambda_2}{v_{p2}}\sin\theta_k^{(4)}=-(\lambda_2+2\mu_2)\frac{\partial U_{k-1}^{(4)}}{\partial z'}\cos\theta_{k-1}^{(4)}, \quad (3)$$

$$U_k^{(1)}\frac{2\mu_1\cos\theta_k^{(1)}}{v_{p2}}+U_k^{(2)}\frac{\mu_1\cos 2\theta_k^2}{v_{p2}\sin\theta_k^{(2)}}+U_k^{(3)}\frac{\mu_2\cos 2\theta_k^{(3)}}{v_{p2}\sin\theta_k^{(3)}}-U_k^{(4)}\frac{\mu_2}{v_{p2}}\cos\theta_k^{(4)}=-\mu_2\frac{\partial U_{k-1}^{(4)}}{\partial z'}\sin\theta_{k-1}^{(4)},$$

where $k=0,1$; the Lamé constants λ_i and μ_i are determined from the equalities $\lambda_i=\rho_i(v_{pi}^2-2v_{si}^2)$, $\mu_i=\rho_i v_{si}^2$; the angles $\theta_k^{(\nu)}$ between the unit vectors $\mathbf{t}_k^{(\nu)}$ and the axis $O'z'$ are defined so that the vectors $\mathbf{t}_k^{(\nu)}$ have positive projections on the axis $O'x'$.

Using formulas (6), (8) of work [5], it is easy to find the intensity $U_0^{(n)}$ of the discontinuity of the highest order of the refracted wave at an arbitrary point of the medium. This quantity has the form

$$U_0^{(n)}(M)=U_0^{(0)}(\mu_0)\frac{\cos\theta_0^4\cdot T(\alpha)}{\sqrt{\cos^2\theta_0^{(4)}+F(l,\alpha^0,\theta_0^{(4)})}},$$

where l is the length of the segment M_0M of the ray of the refracted wave, α is the angle of incidence of the wave $\mathbf{U}^{(0)}(x,y,z,t)$ at the point M_0, smaller than the limiting value $\alpha_0=\arcsin v_{p1}/v_{p2}$ (see Fig. 1); $T(\alpha)$ is the refraction coefficient; the function $F(l,\alpha,\theta_0^{(4)})$, depending on the principal radii of curvature of the front of the incident wave, has the properties

$$F(l,\alpha,\pi/2)\neq 0,\qquad \frac{\partial F(l,\alpha,\pi/2)}{\partial\theta_0^{(4)}}\neq\infty.$$

In view of the fact that $U_0^{(4)}=0$ for $z=0$ ($\alpha=\alpha_0$), corresponding to the point A, from (3) with $k=0$ there follow the relations $U_0^{(1)}=U_0^{(2)}=U_0^{(3)}=0$, from which it is evident that the discontinuities on the fronts of the head waves are one order lower than the discontinuity in the incident wave.

Considering the system (3) for $k=1$, as also for $U_0^{(4)}(M)$, one could determine the quantities $U_1^{(4)}$ and $\mathbf{t}_1^{(4)}$. In this case the system (3) would contain one equation that is a consequence of the others. However, the actual determination of the quantities $U_1^{(4)}$ and $\mathbf{t}_1^{(4)}$ is difficult. It is more convenient to con–

* It should be noted that all τ_ν coincide for $z=0$.

take them as unknowns and introduce one more relation connecting $U_1^{(4)}$ with $U_0^{(4)}$. This relation is easily obtained by projecting the vector equality (4) of work [1] onto the tangential direction (s) to the front in the case of the longitudinal wave $U^{(4)}(x, y, z, t)$ and onto the normal direction (n) in the case of the transverse wave. It has the form

$$U_{1s}^{(4)} = -v_{p2}\frac{\partial U_{0n}^{(4)}}{\partial s}. \tag{4}$$

Considering (4) for $z = 0$ together with (3), we can find

$$U_1^{(v)}(A) = U_0^{(0)}(\widetilde{M}_0)\frac{v_{p2}\Gamma_v(v_p, v_s, \rho)\,\partial\cos\theta_0^{(4)}}{\sqrt{F(l, \alpha, \theta_0^{(4)})}\,\partial z}, \tag{5}$$

whence, expanding the derivative with respect to z, we obtain

$$U_1^{(v)}(A) = U_0^{(0)}(\widetilde{M}_0)\frac{\Gamma_v(v_{p1}, v_{s1}\rho)\sqrt{m^2 r_1 r_2}}{\cos\alpha_0\, l_0^{3/2}\sqrt{l_0 + m r_2}}. \tag{6}$$

In formulas (5) and (6) the following notation has been introduced: $m = v_{p1}/v_{p2}$; r_1 and r_2 are the radii of curvature of the normal sections of the front of the incident wave at the point $\widetilde{M}_0$ (where the wave is incident at the critical angle α_0), respectively in the plane of incidence and in the plane perpendicular to it; l_0 is the distance from the point $\widetilde{M}_0$ to the point A (Fig. 1). The quantities $\Gamma_v(v_p, v_s, \rho)$, called the coefficients of formation of head waves, differ only by simple factors from the coefficients tabulated in [2].

Using (2), for an arbitrary observation point N we finally obtain:

$$U_1^{(v)}(N) = \frac{U_1^{(v)}(A)\sqrt{m r_2 + l_0}}{\sqrt{m r_2 + l_0 + L\sin\beta_v}},$$

where L is the distance AN; β_v is the acute angle between the axis Oz and the normal to the front of the head wave.

In the case considered by us, waves of surface type are absent. They arise if fewer than three fronts of head waves are joined at the point A. The intensities and the parameters x_ν for "surface" waves can be determined from a system analogous to (3).

In conclusion we note that formula (6) remains valid also in the case when only one of the half-spaces (namely, the one in which the body wave producing the

[1] V. M. Babich, DAN, 110, No. 3 (1956).
[2] Materials of quantitative study of the dynamics of seismic waves, 2, L., 1957.

head waves propagates) is homogeneous, while the elastic characteristics of the other depend on z.

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Note: Figure translations are in progress. See original paper for figures.