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MATHEMATICS

1958

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Abstract

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SOME FUNCTIONAL INEQUALITIES OF THE TYPE OF EMBEDDING THEOREMS

(Presented by Academician S. L. Sobolev on VIII 6, 1958)

In the present note we consider certain estimates for functions defined in a domain D of n-dimensional space, analogous to those given by the embedding theorems of S. L. Sobolev ([1,2]), but under some additional assumptions. In contrast to the conditions of S. L. Sobolev' s theorems, in which one and the same estimate is prescribed for integrals of the p-th powers of the l-th derivatives over any subdomain Ω of the domain D, we shall assume this estimate to depend on some positive power α of the diameter either of the whole subdomain Ω, or of its sections of smaller number of dimensions. For the case $(l+\alpha)p > n$, under somewhat different assumptions on the functions, analogous results were obtained by Greco ([3]) and Nirenberg ([4]).

I. With respect to the domain D, in what follows we shall assume that it is such that through each of its points one can draw an n-dimensional spherical sector of constant radius and shape with vertex at this point, lying entirely in D. The class of domains of this kind will be denoted by C_H^n, where $\mathcal{H}$ is the value of the greatest admissible radius of the attaining sector.

Let us introduce the following notation. By D_m we shall denote an arbitrary section of the domain D by the hyperplane $x_{m+1} = \text{const}, \dots, x_n = \text{const}$. Let D_m be some section of the domain D by the hyperplane $x_{m+1} = a_{m+1}, \dots, x_n = a_n$; then by $[D_m]^d_{n-m}$ we shall denote the set of points $P(x_1, \dots, x_m, x_{m+1}, \dots, x_n)$ of the domain D for whose coordinates the inequalities $|x_i - a_i| \le d$, $i = m+1, \dots, n$, hold. In particular, for example, $[D_0]^d_n$ is the part of the domain D contained inside a certain n-dimensional cube with side $2d$, while $[D_n]^d_0 \equiv D$.

II. In what follows we shall assume that $f(x_1, x_2, \dots, x_n)$ is a continuous function defined in the domain $D \in C_H^n$, having continuous derivatives up to order l, and satisfying the conditions

$$1) \qquad \left[\int_{(D)} \cdots \int |f|^p \, dv_n \right]^{\frac{1}{p}} \le A \qquad (p \ge 1), \tag{1}$$

where $dv = dx_1 \cdots dx_n$;

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2) there exists a constant $M > 0$ such that for every integer m, $0 \le m \le n$, and every $d > 0$ the inequality h

$$\max_{D_m} \left[\int\limits_{[D_m]^d_{n-m}} \cdots \int \left(\sum_{i_1,\dots,i_l=1}^{n} \left| \frac{\partial^l f}{\partial x_{i_1} \cdots \partial x_{i_l}} \right|^2 \right)^{\frac{p}{2}} dv_n \right]^{\frac{1}{p}} \le M d^{\alpha m}, \qquad (2)$$

where α_m $(m = 0, 1, \dots, n)$ are fixed numbers satisfying the inequalities

$$\alpha_0 \geqslant \alpha_1 \geqslant \cdots \geqslant \alpha_m \geqslant \cdots \geqslant \alpha_n = 0, \qquad \alpha_m \leqslant \frac{n-m}{p}. \qquad (3)$$

From the fact that $\alpha_n = 0$, it follows that for $d \geqslant 1$ the term d^{α_m} on the right-hand side of inequality (2) may be omitted.

Theorem 1. Let $f(x_1, \dots, x_n)$ satisfy conditions (1)–(3); let k be an integer, with $0 \leqslant k \leqslant l-1$.

Then:

1) if

$$l + \alpha_0 - \frac{n}{p} - k = \varepsilon_0 - k > 0$$

and

$$\frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}}$$

is continuous in

$$\overline{D} = D + \Gamma$$

(Γ is the boundary of D), then for every point $P \in \overline{D}$ the estimate

$$\left| \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}} \right| \leqslant C_1 A h^{-k-\frac{n}{p}} + C_2 M h^{\varepsilon_0 - k}, \qquad (4)$$

holds, where

$$\varepsilon_0 = l + \alpha_0 - \frac{n}{p};$$

2) if

$$l + \alpha_0 - \frac{n}{p} - k \leqslant 0,$$

m is an integer, $1 \leqslant m \leqslant n$,

$$l + \alpha_m - \frac{n-m}{p} - k > 0,$$

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then for any section D_m of the domain D and any exponent $q \geqslant p$ such that

$$q < \frac{m - (\alpha_0 - \alpha_m)p}{n - (l - k + \alpha_0)p}\, p, \tag{5}$$

the inequality

$$\left[\underbrace{\int \cdots \int}_{(D_m)} \left| \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}} \right|^q dv_m \right]^{\frac{1}{q}} \leqslant C_3 A h^{\frac{m}{q} - k - \frac{n}{p}} + C_4 M h^{\varepsilon_m - k}, \tag{6}$$

holds, where $dv_m = dx_1 \cdots dx_m$;

$$\varepsilon_m = l + \alpha_0 \left(1 - \frac{p}{q}\right) + \alpha_m \frac{p}{q} + \frac{m}{q} - \frac{n}{p};$$

h is an arbitrary positive number $\leqslant \mathcal{H}$; C_1, C_2, C_3, C_4 are constants not depending on A, M, h, a, but only on $l, k, \alpha_0, \alpha_m, p, q, m, n$ and on the form of the domain D (on the solid angle of the sector reaching each point of the domain D).

From inequalities (4) and (5) the following inequalities also follow:

$$\left| \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}} \right| \leqslant C_5 \left(A^{\varepsilon_0 - k} M^{k + \frac{n}{p}} \right)^{\frac{1}{l + \alpha_0}} + C_6 A, \quad \text{if } \varepsilon_0 - k > 0; \tag{4'}$$

$$\left[\underbrace{\int \cdots \int}_{(D_m)} \left| \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}} \right|^q dv_m \right]^{\frac{1}{q}} \leqslant$$

$$\leqslant C_7 \left(A^{\varepsilon_m - k} M^{k + \frac{n}{p} - \frac{m}{q}} \right)^{\frac{1}{l + \alpha_0 (1 - \frac{p}{q}) + \alpha_m \frac{p}{q}}} + C_8 A, \tag{6'}$$

if

$$\varepsilon_0 - k \leqslant 0, \qquad l + \alpha_m - \frac{n - m}{p} - k > 0, \qquad p \leqslant q < \frac{m - (\alpha_0 - \alpha_m)p}{n - (l + \alpha_0 - k)p}\, p.$$

It is not difficult to show by examples that the indicated estimates are sharp in order.

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Fig. 1

Figure 1: Fig. 1

III. Suppose now that the domain $D \in C^n_{\mathcal{H}}$ is such that any two of its points P and Q can be joined by a broken line $P_0 P_1 \ldots P_{s-1} P_s$ ($P_0 = P$; $P_s = Q$) with a finite number of links (the number of links for any two points P and Q is bounded and $\leq s_0$) and with lengths not exceeding $|Q - P|$, and such that for any two adjacent vertices P_{i-1} and P_i there exist reaching sectors of one and the same aperture and radius, whose axes lie on the segment $P_{i-1}P_i$, and arranged so that the spherical surface of one of them lies inside the other and conversely (see Fig. 1).

Theorem 2. *Let the domain D satisfy the conditions of the present item; P, Q are two arbitrary points of D. Then under the conditions of item 1) of Theorem 1, for any number λ satisfying the inequalities $0 < \lambda \leq \varepsilon_0 - k$, $\lambda \leq 1$, the estimate*

$$\frac{\left|\frac{\partial^k f(P)}{\partial x_{i_1} \ldots \partial x_{i_k}} - \frac{\partial^k f(Q)}{\partial x_{i_1} \ldots \partial x_{i_k}}\right|}{|Q-P|^\lambda} \leq \begin{cases} C_9 \left(A h^{-k-\frac{n}{p}-\lambda} + M h^{\varepsilon_0 - k - \lambda} \right), & \text{if } \varepsilon_0 - k > 1 \text{ or } \varepsilon_0 - k < 1; \\ C_{10} \left(A h^{-k-\frac{n}{p}-\lambda} + \dfrac{1}{1-\lambda} M h^{\varepsilon_0 - k - \lambda} \right), & \text{if } \varepsilon_0 - k = 1,\ \lambda < 1; \\ C_{11} \left[A \mathcal{H}^{-\frac{n}{p}-k-1} + M \left(1 + \left|\ln \frac{\mathcal{H}}{H}\right| \right) \right], & \text{if } \varepsilon_0 - k = 1,\ \lambda = 1, \end{cases} \tag{7}$$

holds, where C_9, C_{10}, C_{11} are constants independent of A, M, h, λ; h is an arbitrary positive number $\leq \mathcal{H}$; $H \leq \mathcal{H}$, $|Q - P|$.

Let S_m be a plane m-dimensional manifold contained in D_m; let $\mathbf{R}$ be a vector such that all vectors $h\mathbf{R}$ ($0 \leq h \leq 1$) translate S_m within D. Such a vector $\mathbf{R}$ will be called **admissible** for $S_m \subset D_m$.

Suppose further that there exist admissible vectors $\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_t$, translating successively S_m into $S_m^{(1)}$, $S_m^{(1)}$ into $S_m^{(2)}, \ldots, S_m^{(t-1)}$ into $S_m^{(t)} = S_m$, such that:

1) $\mathbf{R}_1 + \ldots + \mathbf{R}_t = \mathbf{R}$ ($t \leq t_0$);
2) $|\mathbf{R}_i| \leq |\mathbf{R}|$, $\quad i = 1, 2, \ldots, t$;
3) for each point P_{i-1} on $S_m^{(i-1)}$ and the corresponding point P_i ($P_i = P_{i-1} + \mathbf{R}_i$) on $S_m^{(i)}$, there exist reaching sectors lying entirely in D and possessing the same property as in III (see Fig. 1); the radii and apertures of these sectors are assumed to be the same.

Theorem 3. *Let $f(x_1, \ldots, x_n)$ satisfy the conditions of item 2) of Theorem 1; $S_m \subset D_m$; $\mathbf{R}$ is a vector admissible for S_m, and with respect to S_m and $\mathbf{R}$ the assumptions made above hold.*

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Then for any $q \geq p$ satisfying condition (5), and for any number λ satisfying the inequalities $0 < \lambda \leq \varepsilon_m - k,\ \lambda \leq 1$, *the estimate*

$$\frac{\left[\underbrace{\int \overset{m}{\cdots} \int}_{(S_m)} \left| \frac{\partial^k f(P+R)}{\partial x_{i_1} \cdots \partial x_{i_k}} - \frac{\partial^k f(P)}{\partial x_{i_1} \cdots \partial x_{i_k}} \right|^q dv_P \right]^{1/q}}{|R|^\lambda} \leq \begin{cases} C_{12} \left[A h^{\frac{m}{q} - \frac{n}{p} - k - \lambda} + M h^{\varepsilon_m - k - \lambda} \right], & \text{if } \varepsilon_m - k > 1 \text{ or } \varepsilon_m - \\ C_{13} \left[A h^{\frac{m}{q} - \frac{n}{p} - k - \lambda} + \dfrac{1}{1-\lambda} M h^{\varepsilon_m - k - \lambda} \right], & \text{if } \varepsilon_m - k = 1,\ \lambda < 1; \\ C_{14} \left[A \mathcal{H}^{\frac{m}{q} - \frac{n}{p} - k - \lambda} + M \left(1 + \left| \ln \dfrac{\mathcal{H}}{H} \right| \right) \right], & \text{if } \varepsilon_m - k = 1,\ \lambda = 1, \end{cases} \tag{8}$$

where C_{12}, C_{13}, C_{14} do not depend on A, M, h, λ; h is an arbitrary positive number $\leq \mathcal{H}$; $H \leq \mathcal{H}, |R|$.

From (7) and (8) follow inequalities analogous to inequalities (4′) and (6′).

IV. The theorems stated will also be valid in the case where condition (2) is replaced by the following:

There exists $M > 0$ such that

$$\max_{D_m} \left[\underbrace{\int \overset{n}{\cdots} \int}_{(D)} r_{n-m}^{-k_{n-m}} \left(\sum_{i_1 \ldots i_l = 1}^{n} \left| \frac{\partial^l f}{\partial x_{i_1} \cdots \partial x_{i_l}} \right|^2 \right)^{p/2} dv_n \right]^{-1/p} \leq M \qquad (m = 0, 1, \ldots, n); \tag{2'}$$

where r_{n-m} is the distance from the plane D_m, $0 \leq k_{n-m} < n - m$, $k_0 = 0$.

In addition, we note that the estimates indicated for functions with continuous derivatives extend also to functions having generalized derivatives in the sense of S. L. Sobolev.

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Received
28 VIII 1958

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