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Abstract

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ON SPACES NUCLEAR IN VARIOUS SENSES

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1. In the note ([1]), D. A. Raikov, comparing the definitions of nuclear spaces proposed by A. Grothendieck in ([3]) and by I. M. Gel' fand in ([5]), showed that in the case of barrelled spaces (*espaces tonnellés*) nuclearity in the sense of Grothendieck implies nuclearity in the sense of Gel' fand.

In the present note two classes of barrelled spaces are indicated for which the converse assertion is also true. In addition, examples are given of spaces nuclear in the sense of Grothendieck but not nuclear in the sense of Gel' fand.

Let us recall the basic definitions.

Definition 1 ([1]). A locally convex space E is called **nuclear in the sense of Grothendieck** ((G)-nuclear) if, for every seminorm p on it, there is a seminorm q such that the mapping of the $(\mathfrak{B})$-space E_q, generated by the seminorm q, into the $(\mathfrak{B})$-space E_p, generated by p, is nuclear.

Definition 2 ([1]). A locally convex space E is called **nuclear in the sense of Gel' fand** ((Γ)-nuclear) if every weakly absolutely convergent series of linear functionals on E converges absolutely with respect to some seminorm of the space E.

Definition 3 ([4]). A locally convex space is called a (DF)**-space** if it contains a fundamental sequence of bounded sets and if, in the strong conjugate to it, every bounded set representable as a countable union of equicontinuously continuous sets is equicontinuously continuous.

The following proposition of Grothendieck will be basic for what follows (([3]), Ch. II, Theorem 8, Corollaries 1 and 2).

A space of type (F) or (DF) is (G)-nuclear if and only if every commutatively (i.e. unconditionally) convergent series of its elements converges absolutely (the latter means absolute convergence with respect to every seminorm).

Lemma. *In the strong conjugate E' to a (Γ)-nuclear space E, every commutatively convergent series converges absolutely.*

Indeed, from the strong commutative convergence of such a series there follows its weak commutative convergence, i.e. weak absolute convergence as a series of functionals. By virtue of Definition 2, the series converges absolutely in the strong conjugate to the space E_p of type $(\mathfrak{B})$, generated by some seminorm on E. And since the mapping $E'_p \to E'$ is continuous, this series converges absolutely in E'.

Corollary 1. *The strong conjugate to a* (Γ)*-nuclear space which is a space of type* (F) *or* (DF) *is* (G)*-nuclear.*

Corollary 2. *A* (Γ)*-nuclear space with Banach strong conjugate is finite-dimensional.*

This corollary follows from the finite-dimensionality of a nuclear Banach space.

2. **Theorem.** In the classes of spaces (F) and of complete spaces (DF), (G)-nuclearity is equivalent to (Γ)-nuclearity.

Proof. 1) Since the strong duals of spaces (F) and (DF) are respectively spaces (DF) and (F) ([4]), it follows, by Corollary 1, that the strong duals of the spaces under consideration, nuclear in the sense of Gelfand, are (G)-nuclear. But spaces (F) and complete spaces (DF) whose strong duals are (G)-nuclear are nuclear in the sense of Grothendieck $((^3), Ch.II, Theorem 7)$.

2) (G)-nuclear (F)-spaces are (Γ)-nuclear, since they are barrelled ([1]).

3) Let E be a complete (G)-nuclear space of type (DF). E is semireflexive. Therefore the set of elements of a weakly absolutely convergent series of functionals on E is strongly bounded. Being at the same time countable, it is, by virtue of Definition 3, equicontinuous. Consequently, there is a seminorm p with respect to which all these functionals are bounded. By the method used in ([1]), the absolute convergence of the series formed from them is established with respect to the seminorm q indicated in Definition 1.

Let us at the same time verify the barrelledness of E. Since E' is a (G)-nuclear (F)-space, every bounded set in it contains a dense countable subset. But since E is a (DF)-space, countable subsets bounded in E' are equicontinuous, so that all bounded sets are equicontinuous. Thus E is quasibarrelled, and since it is semireflexive, it is barrelled.

Thus, the classes mentioned in the theorem do not go beyond the limits of barrelled spaces.

3. The examples that will be given in this section show that there exist (G)-nuclear spaces, even complete ones, whose strong duals are infinite-dimensional $(\mathfrak{B})$-spaces.

By virtue of Corollary 2 these spaces **cannot** be (Γ)-nuclear. The simplest example is probably that of the weak dual of a $(\mathfrak{B})$-space. It is (G)-nuclear as a subspace of a product of lines $((^3), Ch.II, Theorem 9)$, and its strong dual will be the original $(\mathfrak{B})$-space.

More interesting is an example of a **complete** space possessing the indicated property. Its construction follows the scheme developed in ([2]).

Call a sequence $\{x_n\}$ of elements of a locally convex space **rapidly decreasing** if the series

$$\sum_{n=1}^{\infty} n^k x_n$$

converges absolutely for every k. In a space B of type $(\mathfrak{B})$, to each rapidly decreasing sequence of its elements $\{x_n\}$ assign the Banach space $B\{x_n\}$, spanned by the closed absolutely convex hull $\Gamma\{x_n\}$ of this sequence in B, with unit ball $\Gamma\{x_n\}$. As in ([2]), we verify the possibility of representing B as an inductive limit of the spaces $B\{x_n\}$ with the natural embeddings.

Let $\{x_n\}$ be a rapidly decreasing sequence. The sequence $\{|x_n|^{-1/2}x_n\}$ also decreases rapidly in B, and moreover the norm of x_n in $B\{|x_n|^{-1/2}x_n\}$ does not exceed $|x_n|^{1/2}$, so that $\{x_n\}$ is a rapidly decreasing sequence in $B\{|x_n|^{-1/2}x_n\}$. The latter is sufficient $((^3), Ch.II, Proposition 4, Corollary 2)$ for the nuclearity of the mapping $B\{x_n\}$ into $B\{|x_n|^{-1/2}x_n\}$.

Since nuclear mappings are completely continuous, the inductive limit of the spaces $B\{x_n\}$ belongs to those studied in ([2]), and therefore B is the strong dual of the projective limit of the $(\mathfrak{B})$-spaces dual to $B\{x_n\}$, with respect to the adjoint mappings. Owing to the nuclearity of the mapping adjoint to a nuclear one, this projective limit gives a complete (G)-nuclear space.

Remark. Since every Banach space is representable

an inductive limit of $(\mathfrak{B})$-spaces with respect to nuclear mappings, then an arbitrary inductive limit of Banach spaces also admits the same representation. With the aid of ([2]) we conclude from this that the *class of strong duals of complete (G)-nuclear spaces coincides with the class of separated inductive limits of Banach spaces with continuous embeddings.*

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CITED LITERATURE

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[3] A. Grothendieck, *Mem. Am. Math. Soc.*, **16**, (1955).
[4] A. Grothendieck, *Summa Bras. Math.*, **3**, 6, 57 (1954).
[5] I. M. Gelfand, *Uspekhi Mat. Nauk*, **11**, 6, 3 (1956).

Note: Figure translations are in progress. See original paper for figures.