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Abstract

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DESCRIPTION OF A CERTAIN CLASS OF REPRESENTATIONS OF THE LORENTZ GROUP

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Irreducible linear representations of the Lorentz group have been completely described in (^{1}). However, since the Lorentz group is noncompact, it has representations that do not decompose into irreducible ones. Therefore there arises the problem of describing all (up to equivalence) linear representations, for example all bounded representations in Banach spaces. In the present note a partial solution of this problem is given, namely, a certain, quite broad, class of representations, called below the class K_0, is described. In the exposition we use the notation and results of (^{1}).

Below we shall have in mind representations $a \to T_a$ of the group $\mathfrak{A}$ in reflexive Banach spaces satisfying the conditions: 1) the operators T_a are bounded for every fixed a; 2) $(T_a\xi, \varphi)$ is continuous in a for all $\xi \in R,\ \varphi \in R'$.

1. **Canonical model.** Let R_0 be a complex reflexive Banach space and $\varepsilon \to A_\varepsilon$ some representation of the subgroup E, given in R_0 (i.e. a one-parameter group of operators in this space). Further, let m be an integer. From these data (i.e. from R_0, A_ε, and m) we define a representation of the group $\mathfrak{A}$ in the following way.

The space of the representation is $R = L_m[R_0]$—the set of all measurable vector-functions $f(u)$, $u \in \mathfrak{u}$, with values in R_0, satisfying the conditions

$$f(\gamma u) = e^{im\omega} f(u) \quad \text{for } \gamma = \begin{pmatrix} e^{-i\omega} & 0 \\ 0 & e^{i\omega} \end{pmatrix}; \qquad \int |f(u)|^2\, du < \infty.$$

The operations in $L_m[R_0]$ are defined in the usual way; the norm

$$|f| = \left(\int |f(u)|^2\, du \right)^{1/2}.$$

The space conjugate to $L_m[R_0]$ is $L_m[R_0']$. Define in $L_m[R_0]$ the representation $a \to T_a$ of the group $\mathfrak{A}$ in the following way:

$$T_a f(u) = \varepsilon_{22}'^{-2} A_{\varepsilon'} f(u\bar{a}) \quad \text{for } ua = \varepsilon' \zeta(u\bar{a}),\ \varepsilon' \in E,\ \zeta \in Z,\ u\bar{a} \in \mathfrak{u}. \qquad (1)$$

The lowest weight k_0 (the least of the weights k for which $\mathfrak{M}_j^k \neq (0)$) is $k_0 = |m|/2$. In the conjugate representation the role of A_ε is played by the operator

$$\hat{A}_\varepsilon = A'_{\varepsilon^{-1}}$$

(the prime denotes the conjugate operator).

All representations $S_{m,\rho}$ described in (^{1}) are contained in this model (when R_0 is one-dimensional and $A_\varepsilon = \varepsilon_{22}^{2\alpha}(\varepsilon) = \varepsilon_{22}^{i\rho}$).

2. **Structure of the group ring.** We formulate a preliminary result, on the basis of which the class of representations under study will be singled out.

The well-known analogue of the Plancherel formula (^{2}) maps the space $L_2(\mathfrak{A})$ onto the Hilbert space H, consisting of infinite matrices*

$$\lambda = \lambda(\chi) = \left\| \lambda_{j_1 j_2}^{k_1 k_2}(\chi) \right\|, \qquad \sum_{\substack{k_1, k_2 \\ j_1, j_2}} \sum_m \int \left| \lambda_{j_1 j_2}^{k_1 k_2}(m,\rho) \right|^2 (m^2 + \rho^2)\, d\rho < \infty,$$

with scalar product

$$(\lambda, \mu) = \sum_{\substack{k_1, k_2 \\ j_1, j_2}} \sum_m \int \lambda_{j_1 j_2}^{k_1 k_2}(m,\rho)\, \overline{\mu_{j_1 j_2}^{k_1 k_2}(m,\rho)}\, (m^2 + \rho^2)\, d\rho.$$

The matrix index is $\binom{k}{j}$, $k = 0, 1, 2, \dots,\ |j| \leq k$; the elements $\lambda_{j_1 j_2}^{k_1 k_2}(\chi)$ are functions of the character $\chi = (m, \rho)$; they differ from zero only when $|m|/2 \leq \min(k_1, k_2)$.

Under this mapping, the image of the ring $X \subset L_2(\mathfrak{A})$ is the ring $\Lambda = \{\lambda\}$ with ordinary matrix multiplication and Hermitian conjugation as involution. In order to construct representations of the ring Λ, it is necessary to know the class of functions $\lambda_{j_1 j_2}^{k_1 k_2}(\chi)$ as functions of the character χ.

Theorem 1. *The ring* Λ *consists of all matrices* $\lambda \in H$ *whose elements* $\lambda_{j_1 j_2}^{k_1 k_2}(\chi)$ *satisfy the conditions:*

1) $\lambda_{j_1 j_2}^{k_1 k_2}(\chi) = \lambda_{j_1 j_2}^{k_1 k_2}(m,\rho)$ are analytically continuable to the whole complex plane $\rho = \rho_1 + i\rho_2$, where they are entire functions of integral order of growth equal to one;**

$$\varkappa^{k_1}_{m\rho}\varkappa^{k_2}_{-m,-\rho}\lambda^{k_1k_2}_{j_1j_2}(m,\rho) = \varkappa^{k_1}_{-m,-\rho}\varkappa^{k_2}_{m\rho}\lambda^{k_1k_2}_{j_1j_2}(-m,-\rho); \tag{2}$$

$$\varkappa^{k_1}_{m,il}\varkappa^{k_2}_{-m,-il}\lambda^{k_1k_2}_{j_1j_2}(m,il) = \varkappa^{k_1}_{l,im}\varkappa^{k_2}_{-l,-im}\lambda^{k_1k_2}_{j_1j_2}(l,im),$$

3)

$$m, l = -k, -k+1, \dots, k, \qquad k = \min(k_1, k_2); \tag{3a}$$

$$\lambda^{k_1k_2}_{j_1j_2}(m,il) = 0, \quad \text{if} \begin{cases} \text{either } k_2 < |l/2| \le k_1, & l > 0, \\ \text{or } k_1 < |l/2| \le k_2, & l < 0, \end{cases} \tag{3b}$$

where it is denoted

$$\varkappa^k_{m\rho} = \prod_{\mu=|m/2|}^{k} \frac{2\mu + i\rho}{\sqrt{4\mu^2+\rho^2}}.$$

For elements $x \in X^k_j$, only the function $\lambda^{kk}_{jj}(m,\rho) = \lambda(m,\rho)$ differs from zero, and conditions (2), (3) in this case take the form

$$\lambda(m,\rho) = \lambda(-m,-\rho); \qquad \lambda(m,il) = \lambda(l,im), \quad |m| \le 2k, \quad |l| \le 2k.$$

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* We have in mind the kernels $K(u_1,u_2;m,\rho)$ in the basis of functions $f^k_j(u) = \sqrt{2k+1}\,c^k_{\nu j}(u)$ with $\nu = m/2$.

** That is,

$$\int_{-\infty+i\rho_2}^{\infty+i\rho_2} |P(\rho_1+i\rho_2)\lambda(m,\rho_1+i\rho_2)|^2 \, d\rho_1 \le C_1 e^{C|\rho_2|},$$

where $P(\rho)$ is an arbitrary polynomial in ρ, and the constants C_1 and C are their own for each function λ and each polynomial P.

If the representation ϑ is reducible and, in one of the invariant subspaces, induces the representation γ, we shall say that γ is contained in ϑ. As is known (^{2}), the representations $S_{m,\rho}$ are reducible if $\rho^2 = -(|m|+2n)^2$, $n = 1, 2, \dots$.

The relations (3) have the following consequence:

I. *A reducible representation* $S_{m,\rho}$ *contains only one irreducible representation: a spinor representation, if* $\operatorname{Im}\rho > 0$*, and an infinite-dimensional one (namely* $S_{i\rho,im}$*), if* $\operatorname{Im}\rho < 0$.

The irreducible (infinite-dimensional) representations $S_{m,il}$, $l = 0, \pm 1, \pm 2, ...$, $m = |l| + 1, |l| + 2, ...$, will be called *special.*

Definition of the class K_0. A representation of the group $\mathfrak{A}$ belongs to the class K_0 if neither it nor its conjugate contains any special or spinor representations.

3. **Main result.**

Theorem 2. *Every representation of the class K_0 decomposes into a discrete direct sum of representations, each of which is equivalent to one of the representations of the canonical model; moreover, in this model the operators A_ε and A'_ε have no eigenvalues ε^n_{22}, where n is an integer of the same parity as m, $|n| \neq |m|$.*

Idea of the proof. A representation of the group $\mathfrak{A}$ is called normal (M. A. Naimark) if the linear span of vectors of the form $\xi = T_x\eta$, where η runs through $\mathfrak{M}^{k_0}_{k_0}$ and x runs through the whole ring X, is dense in R, and the analogous set of vectors is dense in R'. All irreducible representations are normal.

M. A. Naimark proved the following proposition:

II (equivalence criterion). *Two normal representations of the ring X with the same lowest weight k_0 are equivalent if and only if the representations of the ring $X^{k_0}_{k_0}$ induced in the subspaces $\mathfrak{M}^{k_0}_{k_0}$ are equivalent.*

Thus, the study of representations of the commutative ring $X^{k_0}_{k_0}$ leads to a description of all normal representations.

III. *Representations given by the model (1) are normal only in the case when the operators A_ε and $A'_{\varepsilon^{-1}}$ have no eigenvalues ε^{2r}, $p = k_0 + 1, k_0 + 2, ...$ The normal representations corresponding to the pairs (m, A_ε) and $(-m, A'_{\varepsilon^{-1}})$ are equivalent.*

IV. *If a normal representation belongs to the class K_0, then it is equivalent to one of the representations of the canonical model.*

(The proof is analogous to that indicated in (1) for irreducible representations.)

In order to describe all representations of the class K_0, define in the ring X a two-sided ideal $\widetilde{X}$ by means of the following set of conditions:

$$\lambda_x(m, -il) = 0, \qquad l = 0, \pm 1, \pm 2, \dots, \qquad m = |l| + 1, |l| + 2, \dots$$

To every irreducible representation γ there corresponds a maximal ideal—the kernel of the homomorphism γ. The ideal $\widetilde{X}$ is the intersection of all maximal ideals corresponding to special representations.

V. *In the case of representations of the class K_0, the linear span of vectors $\xi = T_x\eta$, where x runs through $\widetilde{X}$ and η through all subspaces $\mathfrak{M}^{k_1}_{j_1}$, is dense in R, and the analogous set of vectors is dense in R'.*

The ideal $\widetilde{X}$ has a simpler structure than the ring X: it contains no relations (3b) between the functions $\lambda^{k_1k_2}_{j_1j_2}(m,\rho)$. It decomposes into a discrete—

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* With a proper definition of the direct sum of Banach spaces (see (^{2}), p. 74).

the direct sum of ideals $\tilde{X}_m$:

$$\tilde{X}_m = \{x : \lambda_x(m', \rho) = 0 \text{ for } |m'| \neq |m|\}.$$

From this fact follows the unique decomposability of vectors of the form $\xi = T_x\eta$, $x \in \tilde{X}$, into components $\xi_m = T_{x_m}\eta$, $x_m \in \tilde{X}_m$, and, taking Proposition V into account, it follows that:

VI. *Every representation of class K_0 decomposes into a discrete direct sum of normal representations (which also belong to the class K_0).*

This completes the proof of Theorem 2. Let us note that one can construct a single model for representations of class K_0—without decomposing them into normal ones.

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REFERENCES

[1] M. A. Naimark, *Dokl. Akad. Nauk SSSR*, **97**, No. 6, 969 (1954).
[2] M. A. Naimark, *Linear Representations of the Lorentz Group*, 1958.

Note: Figure translations are in progress. See original paper for figures.

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