MATHEMATICS

I. A. Bakhtin and M. A. Krasnosel' skii

1958

SovietRxiv

Abstract

Full Text

MATHEMATICS

I. A. Bakhtin and M. A. Krasnosel' skii

On the Theory of Equations with Concave Operators

(Presented by Academician P. S. Aleksandrov on 9 VI 1958)

1. Convergence of the sequence

$$\varphi_{n+1} = A\varphi_n \quad (n = 0, 1, 2, ...) \tag{1}$$

to the solution φ^* of the equation $\varphi = A\varphi$ is usually a consequence of the fact that the nonlinear operator A is a contraction operator in some function space. In other cases the convergence of the sequence (1) follows from its boundedness and monotonicity; in these cases the main difficulty consists in finding an initial approximation φ_0 that is "less" (or "greater") than the unknown solution φ^*.

In the present paper we study the question of the applicability of the method of successive approximations for constructing solutions of equations with concave operators, the theory of which was developed in ([2–5]).

2. Let E be a real Banach space in which two cones K and K_1 are given, with $K \subset K_1$. We introduce in E a semi-ordering by means of the "larger" cone K_1: we shall write $x \preccurlyeq y$ if $y - x \in K_1$.

Everywhere below it is assumed that for any elements $x, y \in K$, from $x \preccurlyeq y$ it follows that $\|x\| \leq m\|y\|$. This property is obviously fulfilled if the cone K_1 is normal in the sense of M. G. Krein (see ([1])). In what follows, u_0 denotes some fixed nonzero element of K.

Let the nonlinear operator A be defined on the "smaller" cone K and leave this cone invariant: $AK \subset K$. Let A be monotone on the cone K in the sense of the introduced semi-ordering: from $x \preccurlyeq y$ it follows that $Ax \preccurlyeq Ay$. Suppose that for every nonzero $x \in K$ there exist numbers $\alpha, \beta > 0$ such that $\alpha u_0 \preccurlyeq Ax \preccurlyeq \beta u_0$. Finally, suppose that for $0 < t < 1$ the inequalities $Atx \succeq tAx$, $Atx \neq tAx$ hold for all such elements $x \in K$ that $x \succeq \gamma u_0$, where $\gamma > 0$. Such operators A will be called **concave**.

A concave operator A will be called $\{K_1, u_0\}$-concave if, for any pair of elements $x, y \in K$ such that $x \succeq \gamma u_0$, $y \succeq \gamma u_0$ $(\gamma > 0)$, $y - x \in K_1$, from $tx \preccurlyeq y$ $(tx \neq y,\ t > 0)$ it follows that $Ay - tAx \succeq \delta u_0$, where $\delta > 0$.

The class of $\{K_1, u_0\}$-concave operators was introduced for consideration in ([5]); this class contains the previously studied classes of u_0-concave and u_0-monotone operators ([2,3]).

The following general assertion holds.

Theorem 1. *Let the concave operator A be completely continuous, and suppose the equation $\varphi = A\varphi$ has in the cone K a unique nonzero solution φ^*. Then the successive approximations (1) converge in norm to φ^* for any $\varphi_0 \in K$, $\|\varphi_0\| \neq 0$.*

Conditions for the existence and uniqueness of solutions for equations with u_0-concave and u_0-monotone operators are given in ([2,3]), and for equations with $\{K_1, u_0\}$-concave operators in ([5]).

3. P. S. Uryson in ([6]) studied the question of positive solutions of the equation $\varphi = A\varphi$, where

$$A\varphi = \int_a^b K[s, t, \varphi(t)]\, dt + f(s). \tag{2}$$

P. S. Uryson' s basic assumptions on the function $K(s,t,u)$ $(a \leqslant s, t \leqslant b,\ 0 \leqslant u < \infty)$ are that $K(s,t,0) \equiv 0$, $K(s,t,u)$ and $K'_u(s,t,u)$ are positive for positive u, and $K'_u(s,t,u)$ decreases as u increases. It is assumed that $f(s) \geqslant 0$. In ([6]), in particular, it is shown (and this is one of the principal stages of the investigation) that there exist initial approximations $\varphi_0(s)$ for which the sequence (1) converges to a solution of the equation $\varphi = A\varphi$. Methods for the effective construction of the initial approximation $\varphi_0(s)$ are not indicated in ([6]).

Since under P. S. Uryson' s conditions the operator (2) is u_0-concave on the cone of nonnegative functions (see ([2],[3])), it follows from Theorem 1 that there is a somewhat unexpected fact: under the conditions found by P. S. Uryson for the existence of a positive solution of the equation $\varphi = A\varphi$ with operator (2), this positive solution can be obtained as the limit (the convergence being uniform) of the sequence (1) for any nonnegative initial approximation $\varphi_0(s)$ $(\varphi_0(s) \not\equiv 0)$.

We shall give a more general assertion, in whose proof certain propositions established jointly by L. A. Lyadyzhenskii and one of the authors are used.

Theorem 2. *Let the functions $K(s,t,u)$, continuous in u and positive for $u > 0$, and $\Phi(s,t,u) = K(s,t,u)/u$ have the following properties:*

a) $K(s,t,0) \equiv 0$, $K(s,t,u)$ increases monotonically as u increases, $0 \leqslant u < \infty$;

b) for $0 \leqslant u_1 < u_2$

$$\inf_{a \leqslant s,t \leqslant b} [\Phi(s,t,u_1) - \Phi(s,t,u_2)] > 0;$$

c) there exist limits, uniform with respect to s, t, of the function $\Phi(s, t, u)$ as $u \to 0$ and as $u \to +\infty$; the first of these limits is a positive bounded function, and the second is either likewise a positive function or identically zero.*

Let the equation $\varphi = A\varphi$ with operator (2), where $f(s)$ is a nonnegative function, have a positive solution $\varphi^*(s)$.

Then the sequence

$$\varphi_{n+1}(s) = \int_a^b K[s, t, \varphi_n(t)]\, dt + f(s) \qquad (n = 0, 1, 2, \ldots)$$

converges uniformly to $\varphi^*(s)$, whatever nonnegative function $\varphi_0(s)$, not identically zero, may be chosen.

In the assertion of Theorem 2 one may consider, instead of the interval $[a, b]$, an arbitrary set of finite measure. From Theorem 1 follows the convergence of successive approximations to positive solutions of a number of other equations. For example, in ([5]) conditions are given for the concavity of nonlinear integral operators in spaces of vector-functions; by virtue of Theorem 1 these conditions guarantee convergence of successive approximations to positive solutions of systems of integral equations.

4. In some cases the assertion of Theorem 1 can be strengthened. We shall call the u_0-norm of an element $x \in E$ the least of the nonnegative numbers α for which the inequalities $-\alpha u_0 \leqslant x \leqslant \alpha u_0$ hold. This norm $\|x\|_{u_0}$ was used in a number of works of M. G. Krein,

L. V. Kantorovich et al.; this norm is defined, generally speaking, not on all elements of the space E. From the convergence of the sequence $x_n \in E$ in the u_0-norm there follows ordinary convergence (since from $x \leq y$, for $x, y \in K$, by assumption it follows that $\|x\| \leq m\|y\|$); the converse is false–if, for example, $E = L_2$, $u_0(s) \equiv 1$, and the cones K and K_1 coincide with the cone of nonnegative functions, then the u_0-norm is defined on bounded functions and convergence in this norm is equivalent to uniform convergence.

The elements having finite u_0-norm form a space $E^{(0)}$, complete in the u_0-norm. Note that in this case the element u_0 is an interior element of the cone $K_1^{(0)} = K_1 \cap E^{(0)}$.

We shall call a concave operator A u_0-concave (cf. (3)) if, for every $\varphi \in K$ such that $\varphi \geq \gamma u_0$, $\gamma > 0$, and for every interval $[a, b] \subset (0, 1)$, one can indicate an $\eta > 0$ such that

$$At\varphi \geq (1 + \eta)tA\varphi$$

($a \leq t \leq b$). Obviously, u_0-concave operators belong to the class of $\{K_1, u_0\}$-concave operators.

If an operator A, satisfying the conditions of Theorem 1, transforms every sequence of elements convergent in the ordinary norm into a sequence convergent in the u_0-norm, then, obviously, the sequence (1) converges to φ^* not only in the ordinary norm, but also in the u_0-norm.

It turns out that in some cases convergence of the sequence (1) in the u_0-norm can be proved without the assumption of complete continuity of the operator A.

Theorem 3. *Let the equation $\varphi = A\varphi$, with a u_0-concave operator A, have a nonzero solution φ^* in K. Then the successive approximations (1) converge to φ^* in the u_0-norm for any nonzero $\varphi_0 \in K$.*

It is clear that, for convergence in the u_0-norm of the sequence (1), it is sufficient that some iteration of the operator A be a u_0-concave operator.

5. If the space E possesses additional properties, then one can dispense with some of the requirements imposed on the operators under consideration.

Theorem 4. *Suppose that in the space E every bounded monotone sequence converges. Suppose that the continuous operator A is concave and that the equation $\varphi = A\varphi$ has in the cone K a unique nonzero solution φ^*. Then the successive approximations (1) converge to φ^* for any nonzero $\varphi_0 \in K$.*

Let us note that under the conditions of Theorem 3 it is not assumed that the operator A is continuous.

For the existence of a nonzero solution of the equation $\varphi = A\varphi$ with a monotone and, in particular, with a concave operator, it is sufficient that there exist elements $x_0, y_0 \in K$ ($x_0 \neq 0$) such that $x_0 \leq y_0$, $x_0 \leq Ax_0$, $Ay_0 \leq y_0$.

6. In all the assertions formulated above it was assumed that the operator A is defined on the whole cone K. In a number of cases the operator A is defined only on the intersection K_r of the cone K with the ball $\|x\| < r$. The definitions of concavity on K_r, u_0-concavity, and $\{K_1, u_0\}$-concavity are naturally carried over to such operators.

For equations with concave (u_0-concave, $\{K_1, u_0\}$-concave) operators on K_r, various assertions hold analogous to Theorems 1, 3, and 4. Suppose, for example, $\psi \in K_r$, ($\psi \neq 0$) and $A\psi \leq \psi$, and suppose that for $\varphi \in K$, from the inequalities $0 \leq \varphi \leq \psi$ it follows that $\varphi \in K_r$. Then, under the conditions of Theorems 1, 3, and 4, the sequences (1) converge to a solution φ^* of the equation $\varphi = A\varphi$ for any such nonzero $\varphi_0 \in K$ that $0 \leq \varphi_0 \leq \psi$.

As an example of a $\{K_1, u_0\}$-concave operator on some K_r one may take the nonlinear integral operator to the study of which the problem of the longitudinal bending of a hinged rod of variable stiffness leads (see ([3,4,7])).

Let us give one more important example. Consider a nonlinear completely continuous operator A ($A0 = 0$), acting in some neighborhood

the zero of a real Hilbert space H and admitting a representation by Taylor' s formula (see ([3,8])): $A\varphi = B\varphi + C\varphi + D\varphi$, where B is a linear self-adjoint operator, $Ct\varphi = t^k C\varphi$, and D is an operator of order higher than k. Let 1 be a simple eigenvalue of the operator B: $Bh = h$, and let all the remaining eigenvalues be less than 1 in absolute value.

Suppose that $(Ch, h) < 0$. It turns out that then one can indicate cones K and K_1, $K \subset K_1$, such that A on some K_r is a $\{K_1, h\}$-concave operator.

The condition $(Ch, h) < 0$ makes it possible (see ([8])) to describe the location of the spectrum of the operator A in a neighborhood of its first bifurcation point. The corresponding eigenfunctions can be obtained as limits of successive approximations under a special choice of the initial approximations φ_0—this follows from the concavity of A and from the assertions formulated above. The initial approximations may be chosen in the form ρh, where for the admissible values of ρ one can give an upper estimate.

Voronezh State
University

Received
10 V 1958

REFERENCES

[1] M. G. Krein, M. A. Rutman, *Uspekhi Mat. Nauk*, **3**, No. 1, 23 (1948).
[2] M. A. Krasnosel' skii, L. A. Ladyzhenskii, *Tr. Moscow Math. Soc.*, **3** (1954).
[3] M. A. Krasnosel' skii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Moscow, 1956.
[4] I. A. Bakhtin, M. A. Krasnosel' skii, *Dokl. Akad. Nauk SSSR*, **105**, No. 4 (1955).
[5] I. A. Bakhtin, *Dokl. Akad. Nauk SSSR*, **117**, No. 1 (1957).
[6] P. S. Uryson, *Trudy po topologii i drugim oblastyam matematiki*, **1**, 1951, pp. 45-77.
[7] Ya. D. Mamedov, *Dokl. Akad. Nauk SSSR*, **118**, No. 1 (1958).
[8] M. A. Krasnosel' skii, *Dokl. Akad. Nauk SSSR*, **109**, No. 3 (1956).

Note: Figure translations are in progress. See original paper for figures.