UNIQUENESS OF THE SYMMETRIC SOLUTION OF THE PROBLEM OF LARGE DEFLECTIONS OF A SYMMETRICALLY LOADED CIRCULAR PLATE

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Abstract

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UNIQUENESS OF THE SYMMETRIC SOLUTION OF THE PROBLEM OF LARGE DEFLECTIONS OF A SYMMETRICALLY LOADED CIRCULAR PLATE

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In papers ($^{1-3}$) the existence of solutions was proved for problems on large deflections of shallow shells and thin plates; however, questions of uniqueness remained unresolved. In the present note a proof is proposed of the uniqueness of the symmetric solution in the problem of large deflections of a symmetrically loaded circular plate.

The Kármán system of equations in the case of a circular symmetrically loaded plate reduces to the system of ordinary differential equations (^{4}):

$$Av - \frac{\lambda}{2}u^2 = 0; \qquad \frac{1}{r}Au + \lambda uv + \int_0^r q\rho\, d\rho = 0, \tag{1}$$

where

$$A(\;) \equiv -r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r(\;), \tag{2}$$

and the boundary conditions for a rigidly clamped plate in the absence of membrane stresses on the contour take the form:

$$u|_{r=1} = 0; \tag{3a}$$

$$v|_{r=1} = 0; \tag{3b}$$

$$\left.\frac{u}{r}\right|_{r=0} < \text{const}; \qquad \left.\frac{v}{r}\right|_{r=0} < \text{const}. \tag{4}$$

Consider the functional space L_ρ with norm $\left(\int_0^1 \frac{1}{\rho}(u^2+v^2)\,d\rho\right)$ and the space W_ρ, formed by the closure of sufficiently smooth functions satisfying conditions (3) and (4), in the norm

$$\int_0^1 \frac{1}{\rho}\left[(Au)^2+(Av)^2\right]\,d\rho.$$

The following relations can be established:

$$\int_0^1 \frac{1}{r}Au\cdot u\,dr = \int_0^1 -\frac{d}{dr}\frac{1}{r}\frac{d}{dr}ru\cdot u\,dr = -\frac{1}{r}u\cdot\frac{dru}{dr}\bigg|_{r=0}^{r=1} + \int_0^1 \frac{1}{r}\frac{dru}{dr}\cdot\frac{du}{dr}\,dr =$$

$$= \int_0^1 \left(\frac{du}{dr}\right)^2 dr + \frac{1}{2}\int_0^1 \frac{u^2}{r^2}\,dr - \frac{u^2}{2r}\bigg|_{r=0}^{r=1} - u\frac{du}{dr}\bigg|_{r=0}^{r=1}. \tag{5}$$

Finally, using the boundary conditions (3) and (4), we obtain

$$\int_0^1 \frac{1}{r}Au\cdot u\,dr = \int_0^1 \left(\frac{du}{dr}\right)^2 dr + \frac{1}{2}\int_0^1 \frac{u^2}{r^2}\,dr. \tag{6}$$

We multiply the first equation of system (1) by $2v/r$, and the second by u/r, integrate from 0 to 1, and add; then the equality

$$2\int_0^1 \frac{1}{r}Av\cdot v\,dr + \frac{1}{\lambda}\int_0^1 \frac{1}{r}Au\cdot u\,dr = -\int_0^1 \frac{u}{r}\int_0^r q\rho\,d\rho\,dr \tag{7}$$

holds. Hence, analogously to (3), we obtain the a priori estimates

$$\int_0^1 \frac{1}{r}(u^2+v^2)\,dr \le \text{const}. \tag{8}$$

The existence of a solution of equation (1) under the boundary conditions (3)–(4), generally speaking, is obtained as a special case in the paper (^{3}), but it is necessary to carry out the proof once more, since one must obtain the existence of precisely a symmetric solution. From the system (1), (3), (4) we pass to the equivalent system of integral equations

$$v + \frac{\lambda}{r}\int_0^r \rho\int_0^\rho \frac{u^2}{2\xi}\,d\xi\,d\rho - \lambda r\int_0^1 \rho\int_0^\rho \frac{u^2}{2\xi}\,d\xi\,d\rho = 0;$$

$$u+\frac{\lambda}{r}\int_0^r\rho\int_0^\rho\frac{uv}{\xi}\,d\xi\,d\rho-\lambda r\int_0^1\rho\int_0^\rho\frac{uv}{\xi}\,d\xi\,d\rho \tag{9}$$

$$+\frac{1}{r}\int_0^r\rho\int_0^\rho\frac{1}{\xi}\int_0^\xi q\eta\,d\eta\,d\xi\,d\rho-r\int_0^1\rho\int_0^\rho\frac{1}{\xi}\int_0^\xi q\eta\,d\eta\,d\xi\,d\rho=0.$$

For the system (9), using the a priori estimates (8) and the equality (6), by the Schauder–Leray method (5) it is easy to obtain an existence theorem for all $\lambda\in[0,1]$. It can be shown that this solution will also be a solution of system (1) from the space W_ρ.

We pass to the proof of uniqueness of the symmetric solution.

The Hildebrandt–Graves theorem holds:

Let $\Psi(V,\lambda)$ be an operator defined for $V\in E_1$ and $\lambda\in[0,1]$, and let the values $\Psi(V,\lambda)\in E_2$; E_1 and E_2 are Banach spaces. Suppose $\Psi(V_0,\lambda_0)=0$, and at the point V_0,λ_0 the operator Ψ is continuous and has, with respect to V in some neighborhood of the point V_0,λ_0, a continuous partial Fréchet derivative which, at the point V_0,λ_0, is a bounded linear operator having a bounded inverse operator. Then $\Psi(V,\lambda)=0$ has a unique solution with respect to V for every λ from a neighborhood of λ_0 (6).

We shall show that the conditions of Hildebrandt–Graves are fulfilled in the case of system (1). First we prove a lemma:

Lemma. For every solution of the system (9), $v\geq 0$.

Indeed: a)

$$\left.\frac{v}{r}\right|_{r=1}=0;$$

b)

$$\frac{d}{dr}\left(\frac{v}{r}\right)=\frac{2\lambda}{r^3}\int_0^r\rho\int_0^\rho\frac{u^2}{2\xi}\,d\xi\,d\rho-\frac{\lambda}{r}\int_0^r\frac{u^2}{2\xi}\,d\xi=$$

$$=\frac{2\lambda}{r^3}\int_0^r\rho\int_0^\rho\frac{u^2}{2\xi}\,d\xi\,d\rho-\frac{2\lambda}{r^3}\int_0^r\rho\,d\rho\int_0^r\frac{u^2}{2\xi}\,d\xi\leq 0\quad\text{for all } r\in[0,1].$$

From a) and b) we directly obtain that $v\geq 0$.

Let us write system (1) in the following form:

$$P(V,\lambda)=0. \tag{10}$$

$P(V,\lambda)$ is the left-hand side of system (1); $V\equiv(u,v)$. The Fréchet derivative at the point $V_0\equiv(u_0,v_0)$ is equal to

$$P'_{V_0}(V,\lambda) \equiv \left(Av - \lambda u_0 u;\ \frac{1}{\chi} Au + \lambda u_0 v + \lambda v_0 u\right),$$

$$\int_0^1 \frac{1}{\rho} P'_{V_0}(V,\lambda) V\, d\rho = \int_0^1 \left(\frac{1}{\rho} Av \cdot v + \frac{1}{\chi\rho} Au \cdot u\right) d\rho + \lambda \int_0^1 \frac{v_0 u^2}{\rho}\, d\rho.$$

If for u_0, v_0 we take a solution of system (1) (the existence of which was shown above), then, using the inequality $v \geqslant 0$, we obtain:

$$\int_0^1 \frac{1}{\rho} P'_{V_0}(V) V\, d\rho \geqslant \|V\|^2_{L^2_\rho},$$

and, consequently,

$$\left\|[P'_{V_0}]^{-1}\right\| \leqslant C.$$

The remaining conditions of the Hildebrandt-Graves theorem are verified directly.

Thus the uniqueness of the symmetric solution for a circular, symmetrically loaded, rigidly clamped plate has been proved.

Remark 1. v corresponds to the radial force, and the lemma proved above has a simple physical meaning: under the given boundary conditions the radial forces in the plate are tensile.

Remark 2. We considered a rigidly clamped plate in the absence of chain forces on the contour. Let us now examine other types of boundary conditions.

1) In the case of a simply supported plate only the form of the second equation (9) changes, and all the arguments are carried out analogously.

2) In the case when a tensile force $T > 0$ is prescribed on the contour ($v|_{r=1} = T$), system (9) takes the form

$$v + \frac{\lambda}{r} \int_0^r \rho \int_0^\rho \frac{u^2}{2\xi}\, d\xi\, d\rho - \lambda r \int_0^1 \rho \int_0^\rho \frac{u^2}{2\xi}\, d\xi\, d\rho - T = 0;$$

$$u + \frac{\lambda}{r} \int_0^r \rho \int_0^\rho \frac{uv}{\xi}\, d\xi\, d\rho - \lambda r \int_0^1 \rho \int_0^\rho \frac{uv}{\xi}\, d\xi\, d\rho +$$

$$+ \frac{1}{r} \int_0^r \rho \int_0^\rho \frac{1}{\xi} \int_0^\xi q\eta\, d\eta\, d\xi\, d\rho - r \int_0^1 \rho \int_0^\rho \frac{1}{\xi} \int_0^\xi q\eta\, d\eta\, d\xi\, d\rho = 0.$$

From this it is clear that $v \geqslant T$. The further proof is carried out analogously to that set forth above.

3) In the case when all three displacements on the contour are equal to zero, boundary condition (3b) takes the form

$$\frac{dv}{dr} - \sigma v\Big|_{r=1} = 0 \quad (0 < \sigma \leqslant k < 1/2)$$

(see, for example, (4)). Under these boundary conditions we still have $v \geqslant 0$, and the operator $A_\rho v$ is positive definite. The latter is proved analogously to the basic case, using in equality (5) the indicated boundary condition.

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