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# MATHEMATICS

BLAGOVEST SENDOV

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**Abstract**

**Full Text**

MATHEMATICS

BLAGOVEST SENDOV

## ON THE QUESTION OF THE EXPANSION OF REGULARLY MONOTONE FUNCTIONS IN A GONCHAROV SERIES

*(Presented by Academician S. N. Bernstein, 15 VII 1957)*

Recently we proved <sup>(1)</sup>, using the method of Ya. A. Tagamlitskii <sup>(2)</sup>, a theorem on the expansion of a certain class of regularly monotone functions in a Goncharov series. In the present note we shall prove this theorem by classical means and at the same time generalize it.

If one takes an arbitrary sequence  $\varepsilon (\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n, \dots; \varepsilon_n = 1 \text{ or } -1)$ , then there exist functions satisfying the conditions

$$\varepsilon_n f^{(n)}(x) \geq 0, \quad 0 \leq x \leq 1, \quad n = 0, 1, 2, \dots \quad (1)$$

Such are, for example, the Goncharov polynomials corresponding to the nodes

$$x_0, x_1, x_2, \dots, x_n, \dots; \quad x_n = \frac{|\varepsilon_n - \varepsilon_{n+1}|}{2}, \quad (2)$$

which we shall denote by  $P_n(x)$ , so that

$$P_0(x) = \varepsilon_0, \quad P_n(x) = \int_{x_0}^x \tau_0 dt_1 \int_{x_1}^{t_1} \tau_1 dt_2 \cdots \int_{x_{n-1}}^{t_{n-1}} \tau_{n-1} dt_n, \quad n = 1, 2, \dots, \quad (3)$$

where  $\tau_k = \varepsilon_k \varepsilon_{k+1}$ .

Denote by  $K_\varepsilon$  the set of regularly monotone functions satisfying conditions (1), and by  $K_\varepsilon^*$  the set of those functions  $f(x) \in K_\varepsilon$  for which  $f^{(n)}(x_n) = 0$  ( $n = 0, 1, 2, \dots$ ).

It is not difficult to see that, if  $f(x) \in K_\varepsilon$ , then

$$f(x) = \sum_{\nu=0}^{\infty} \varepsilon_\nu f^{(\nu)}(x_\nu) P_\nu(x) + R(x),$$

where  $R(x) \in K_\varepsilon^*$ .

From the works of S. N. Bernstein <sup>(3,4)</sup> it is known that in the case of absolutely monotone functions  $R(x) \equiv 0$ , while in the case of cyclically monotone functions of sine (cosine) type,

$$R(x) = A \sin \frac{\pi}{2} x \quad \left( R(x) = A \cos \frac{\pi}{2} x \right).$$

In the work <sup>(1)</sup> we considered the case when the sequence  $\varepsilon$  is periodic, and established that  $R(x) = AR_\varepsilon(x)$ , where  $A \geq 0$  and  $R_\varepsilon(x)$  does not depend on  $f(x)$ .

**Theorem 1.** *The set  $K_\varepsilon^*$  contains only one function, up to a nonnegative constant factor, if the sequence  $\varepsilon$  contains, an infinite number of times, one and the same combination consisting of  $p \geq 4$  terms of the form*

$$-\varepsilon, \varepsilon, \varepsilon, \dots, \varepsilon, -\varepsilon, \quad \text{where } \varepsilon = \pm 1. \quad (4)$$

This condition will be called condition (A).

In view of the possibility of replacing  $x$  by  $1 - x$  and  $f(x)$  by  $-f(x)$ , we may assume, for definiteness, that  $\varepsilon_0 = \varepsilon_1 = 1$ ,  $x_0 = 0$ . We restrict ourselves to the case  $\varepsilon = 1$  (the case  $\varepsilon = -1$  is considered analogously).

The proof\* of the theorem is based on the following lemmas.

**Lemma 1.** If condition (A) is satisfied, then the sequence (2) contains an unbounded number of zeros and ones.

**Lemma 2.** If

$$f_1(x), f_2(x), \dots, f_n(x), \dots \quad (5)$$

is a sequence of functions belonging to  $K_\varepsilon$ , and  $f_n(1) \leq A$ ,  $n = 1, 2, \dots$ , then from (5) one can extract a subsequence

$$f_{n_1}, f_{n_2}, \dots, f_{n_m}, \dots,$$

for which there exists the limit

$$\lim_{m \rightarrow \infty} f_{n_m}^{(k)}(x) = f^{(k)}(x), \quad k = 1, 2, \dots, \quad x \in [0, 1].$$

We do not give the proof.

It follows from Lemma 2 that  $K_\varepsilon^*$  contains a nonvanishing function  $R(x)$ , normalized by the condition  $R(1) = 1$ . Indeed, in place of (5) it suffices to take the sequence of polynomials

$$\tilde{P}_n(x) = \frac{1}{P_n(1)} P_n(x), \quad \tilde{P}_n(1) = 1.$$

With the aid of Rolle' s theorem, Lemmas 3 and 4 are proved.

**Lemma 3.** If  $f(x)$  belongs to  $K_\varepsilon^*$  and  $f(1) = 1$ , then the equation

$$f^{(k)}(x) - \tilde{P}_n^{(k)}(x) = 0, \quad n = 1, 2, \dots,$$

has one and only one root  $\xi_k \in (0, 1)$  for  $k = 1, 2, \dots, n$ .

**Lemma 4.** If  $f(x)$  and  $g(x)$  belong to  $K_\varepsilon^*$ , and the equation

$$f^{(k)}(x) - g^{(k)}(x) = 0, \quad k = 1, 2, \dots,$$

has a root  $\xi_{k_0} \in (0, 1)$  for some nonnegative integer  $k_0$ , then this equation has a root  $\xi_k \in (0, 1)$  also for every  $k > k_0$ .

Let  $n_1 < n_2 < \dots < n_k < \dots$  be the sequence of indices for which  $\varepsilon_{n_1}, \varepsilon_{n_2}, \dots, \varepsilon_{n_k}, \dots$  are the first terms of the combination (4). Then  $\varepsilon_{n_k+m} = 1$  for  $m = 1, 2, \dots, p-2$ ;  $\varepsilon_{n_k} = \varepsilon_{n_k+p-1} = 1$ ;  $x_{n_k+m} = 0$  for  $m = 1, 2, \dots, p-3$ ;  $x_{n_k} = x_{n_k+p-2} = 1$ . We prove the inequality

$$\frac{p-2}{(p-1)!} P_{n_k}(1) \leq P_{n_k+p-2}(1). \quad (6)$$

Indeed:

$$\begin{aligned} P_{n_k+p-2}(x) &= \frac{1}{(p-3)!} \int_{x_0}^x \tau_0 dt_1 \int_{x_1}^{t_1} \tau_1 dt_2 \dots \int_{x_{n_k-1}}^{t_{n_k-1}} \tau_{n_k-1} dt_{n_k} \int_{t_{n_k}}^1 t^{p-3} dt = \\ &= \frac{1}{(p-2)!} P_{n_k}(x) - \frac{1}{(p-1)!} \int_{x_0}^x \tau_0 dt_1 \int_{x_1}^{t_1} \tau_1 dt_2 \dots \\ &\dots \int_{x_{n_k-1}}^{t_{n_k-1}} \tau_{n_k-1} du \geq \frac{1}{(p-2)!} P_{n_k}(x) - \frac{1}{(p-1)!} P_{n_k}(x). \end{aligned}$$

Putting  $x = 1$ , we obtain (6).

\* The possibility of such a classical proof was pointed out to me by Prof. Ya. A. Tagamlitskii.

**Lemma 5.** If  $f(x) \in K_\varepsilon^*$ , then:

$$I. |f^{(n)}(1-x_n)| \leq |f^{(n+1)}(1-x_{n+1})|, \quad n = 0, 1, 2, \dots$$

$$\text{II. } -f^{(n_k)}(0) \geq \frac{f(1)}{P_{n_k}(1)}.$$

$$\text{III. } f^{(n_k+p-3)}(1) \leq \frac{f(1)}{P_{n_k+p-2}(1)}.$$

**Proof.**

I.

$$|f^{(n)}(1-x_n)| = |f^{(n)}(1-x_n) - f^{(n)}(x_n)| = |1-2x_n| |f^{(n+1)}(\xi)| \leq |f^{(n+1)}(1-x_{n+1})|.$$

II.

$$\begin{aligned} f(1) &= \int_{x_0}^1 \tau_0 dt_1 \int_{x_1}^{t_1} \tau_1 dt_2 \cdots \int_{x_{n_k-1}}^{t_{n_k-1}} \tau_{n_k-1} \varepsilon_{n_k} f^{(n_k)}(t) dt \leq \\ &\leq \varepsilon_{n_k} f^{(n_k)}(0) P_{n_k}(1) = -f^{(n_k)}(0) P_{n_k}(1). \end{aligned}$$

III. If  $f(x)$  is a nonvanishing function, then  $f(1) \neq 0$ . From Lemma 3 it follows that the equations

$$\varphi(x) = \frac{1}{f(1)} f^{(n_k+p-3)}(x) - \widetilde{P}_{n_k+p-2}^{(n_k+p-3)}(x) = 0$$

and

$$\varphi'(x) = 0 \tag{1}$$

have, respectively, roots  $\xi'$  and  $\xi'' \in (0, 1)$ . It is not hard to see that  $0 < \xi'' < \xi'$ , and

$$\widetilde{P}_{n_k+p-2}^{(n_k+p-3)}(x) = \frac{x}{P_{n_k+p-2}(1)}.$$

We assert that  $\varphi(1) \leq 0$ . Indeed, suppose that  $\varphi(1) > 0$ . Then we would have

$$\varphi(1) = \varphi(1) - \varphi(\xi') = (1 - \xi')(\eta' - \xi'')\varphi''(\eta') > 0,$$

where  $1 > \eta' > \xi' > \xi''$ ; consequently,

$$\varphi''(\eta') > 0.$$

But

$$\varphi_x = \frac{1}{f(1)} f^{(n_k+p-1)}(x) \leq 0, \quad x \in [0, 1],$$

i.e.  $\varphi(1) \leq 0$ . On the other hand,

$$\varphi(1) = \frac{1}{f(1)} f^{(n_k+p-3)}(1) - \frac{1}{P_{n_k+p-2}(1)} \leq 0,$$

which proves III.

**Lemma 6.** If  $f(x)$  and  $g(x)$  belong to  $K_\varepsilon^*$  and  $f(1) = 1$ ,

$$g(1) < \frac{p-2}{(p-1)!},$$

then the function  $h(x) = f(x) - g(x)$  also belongs to  $K_\varepsilon^*$ .

**Proof.** From Lemma 5 and (6) we find

$$|g^{(n_k+1)}(1)| \leq |g^{(n_k+p-3)}(1)| \leq \frac{g(1)}{P_{n_k+p-2}(1)} < \frac{p-2}{(p-1)!} \frac{(p-1)!}{p-2} \frac{1}{P_{n_k}(1)} \leq |f^{(n_k)}(0)|,$$

i.e.

$$|g^{(n_k+1)}(1)| < |f^{(n_k)}(0)|. \quad (7)$$

It is not hard to see that

$$-f^{(n_k)}(x) \geq (1-x)|f^{(n_k)}(0)|, \quad -g^{(n_k)}(x) \leq (1-x)|g^{(n_k+1)}(1)|, \quad (8)$$

since  $\varepsilon_{n_k} = -1$ ,  $\varepsilon_{n_k+1} = \varepsilon_{n_k+2} = 1$ . From (7) and (8) it follows that the equation

$$f^{(n_k)}(x) - g^{(n_k)}(x) = 0$$

has no solution in the interval  $(0, 1)$ . Hence, by Lemma 4, it follows that the function  $h(x) = f(x) - g(x)$  is regularly monotone on the segment  $[0, 1]$ , since the sequence  $\{n_k\}$  is unbounded. On the other hand,

$$h^{(n)}(x_n) = 0, \quad n = 0, 1, 2, \dots,$$

and  $h(x) \geq 0$  for  $x \in [0, 1]$ . From this it follows that  $h(x) \in K_\varepsilon^*$ .

As we have already noted, there exists a nonvanishing function  $R_\varepsilon(x) \in K_\varepsilon^*$  for which

$$R_\varepsilon(1) = 1.$$

**Lemma 7.** If  $f(x) \in K_\varepsilon^*$ , then  $f(x) = AR_\varepsilon(x)$ , where  $A = f(1) \geq 0$ .

**Proof.** The function

$$h(x) = R_\varepsilon(x) - cf(x) \in K_\varepsilon^*, \quad (9)$$

if  $0 \leq c < \frac{p-2}{f(1)(p-1)}$ . Denote by  $c_0$  the greatest constant  $c$  satisfying (9).

Such a constant exists by virtue of the closedness of  $K_\varepsilon^*$ . We shall prove that  $R_\varepsilon(x) - c_0f(x) \equiv 0$ . If this is not so, then  $R_\varepsilon(1) - c_0f(1) > 0$ , and, by Lemma 6, there exists a constant  $\rho > 0$  such that  $h_1(x) = h(x) - \rho f(x) \in K_\varepsilon^*$ . But  $h_1(x) = R_\varepsilon(x) - (c_0 + \rho)f(x)$ , which contradicts the choice of  $c_0$ .

Using Lemma 2, one can show that the limit exists

$$\lim_{n \rightarrow \infty} \frac{P_n(x)}{P_n(1)} = R_\varepsilon(x), \quad x \in [0, 1].$$

It is not difficult to see that Lemma 7 is merely another formulation of Theorem 1.

Let us note that condition (A) is satisfied if and only if the sequence of type numbers <sup>(5)</sup>  $\lambda_1, \lambda_2, \dots, \lambda_n, \dots$  contains a bounded partial subsequence.

If the sequence  $\varepsilon$  is periodic and its primitive period  $q \geq 3$ , then the sequence  $\varepsilon$  satisfies condition (A).

In <sup>(1)</sup> it is proved that if  $\varepsilon$  is periodic and  $q$  is its period, then  $R_\varepsilon(x)$  satisfies the differential equation

$$f^{(q)}(x) = \alpha^q f(x), \quad \alpha > 0.$$

This assertion follows from Lemma 7 owing to the fact that, if  $f(x) \in K_\varepsilon^*$ , then also  $f^{(q)}(x) \in K_\varepsilon^*$ .

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*Note: Figure translations are in progress. See original paper for figures.*

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