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Abstract

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ON THE ARCHIMEDEAN PRINCIPLE IN PARTIALLY ORDERED FACTOR-LINEALS

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This note investigates the question of when the factor-lineal X/N of an Archimedean K-lineal X by its normal sublineal N turns out to be an Archimedean K-lineal*.

It is known that if X is an arbitrary K-lineal and N is its normal sublineal, then, identifying all elements of X that belong to one adjacency class modulo N, we again obtain a K-lineal X/N (if in X/N the algebraic operations are defined in the natural way, and the partial order as follows: let $\tilde{x} \in X/N$; we regard $\tilde{x} > 0$ if in $\tilde{x}$ there is at least one $x' \in X$ such that $x' > 0$ and $x' \in N$). This result is contained, for example, in ([3]) (in the exercises to Ch. II, § 1). Here the mapping $X \to X/N$, assigning to each $x \in X$ that class $\tilde{x} \in X/N$ in which x is contained, is a homomorphism with respect to the algebraic and lattice operations.

Let X be an Archimedean K-lineal; then N is also an Archimedean K-lineal; however X/N may fail to be Archimedean. Consider, for example, the factor X/N in the case where X is the extended K-space and $N = B(X)$ is the subspace of its bounded elements. For any real λ, μ we have $\lambda^2+\mu^2 \geq \lambda\mu$. Replacing λ in the inequality by x, and μ by $n1$, and using the property of preservation of relations (([1]), p. 126), we obtain $x^2 + n^2 1 \geq x \cdot n1 = nx$. Therefore, passing in this inequality to the images in X/N, we shall have for any x

$$(\tilde{x}^2) \geq \widetilde{nx} \quad \text{for } n = 1, 2, ... \quad (\text{since } n^2 1 \in N),$$

i.e. in X/N every element turns out to be non-Archimedean.

Theorem 1 gives a necessary and sufficient condition for the factor-lineal X/N to be Archimedean. In Theorems 2–6 this general condition is simplified for various particular types of K-lineals.

Theorem 1. *Let X be an Archimedean K-lineal, and N its normal sublineal. Then, in order that the factor-lineal X/N be Archimedean, it is necessary and sufficient that N satisfy the following condition:*

A_1. *Let $x_n \in N$, $x_n \geq 0$ $(n = 1, 2, ...)$, and let the sequence $\{x_n\}$ be bounded in X. Let, further, the numbers $\lambda_n \geq 0$ and $\lambda_n \to 0$. Then, if $x \in X$ and $0 \leq x' \leq y$ for every y that is an upper bound of the set $\{\lambda_n x_n\}$, then $x \in N$.*

A set X' of a K-lineal X is called a **component** in X if there exists a set $E \subset X$ such that X' is the totality of all elements $x \in X$ disjoint from E. The **component generated by some set** $H \subset X$ is the smallest component in X containing H. In an Archimedean K-lineal the definition of component given here is equivalent to the definition of component in a K-space introduced in ([1]) (p. 62).

Let X' be a component in the K-lineal X, $x \in X$, $x \geq 0$. Then $\sup x'$ over all $x' \in X'$, $x' \leq x$ (if such a supremum exists), is called the **projection** of x in X'.

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* For the definition of a K-lineal and the formulation of the Archimedean principle, see, for example, ([1]).

For an arbitrary $x \in X$, its projection in X' is the difference of the projections in X' of its positive and negative parts.

Theorem 2. Let X be an Archimedean K-lineal in which there exists the projection of each element $y \in X$ onto any component generated by an arbitrary element $x \in X$; let N be its normal sublineal. Then, in order that the factor-lineal X/N be Archimedean, it is necessary and sufficient that N satisfy the following condition:

A_2. Let $x_n \in N$, $x_n \geqslant 0$ $(n = 1, 2, ...)$, $x_n d x_p$ for $n \neq p$, and let there exist $\sup x_n \in X$. Let, further, $\lambda_n \geqslant 0$ and $\lambda_n \to 0$. Then, if $x \in X$ and $0 \leqslant x \leqslant y$ for any y which is an upper bound of the set $\{\lambda_n x_n\}$, then $x \in N$.

Remark. Denote by A' (A'') the weakened condition which is obtained from A_1 if in the latter one considers only pairwise disjoint x_n (x_n for which $\sup x_n$ exists). Then it can be shown that in Theorem 1 the condition A_1 cannot be replaced either by A' or by A'', i.e. in this sense the condition A_1 in Theorem 1 cannot be brought closer to A_2. Namely, let X be the K-lineal of all functions on $[0, 1]$ representable in the form

$$\alpha \left(\frac{1}{t - \frac{1}{4}} + \frac{1}{t - \frac{3}{4}} \right) + \varphi(t),$$

where $\varphi \in C_{[0,1]}$, α is a real number; N is the totality of all functions from X which vanish on $[0, 1/2]$ and for $t = 3/4$. Then one can verify that the normal sublineal N in X satisfies the conditions A_2, A', and A'', but does not satisfy A_1.

From Theorem 2 Theorem 3 immediately follows.

Theorem 3. Let X be a K'-space, N its normal K'-subspace. Then, in order that the factor X/N be Archimedean, it is necessary and sufficient that N satisfy the following condition:

A_3. Let $x_n \in N$, $x_n \geqslant 0$ $(n = 1, 2, ...)$, $x_n d x_p$ for $n \neq p$, and let $\sup x_n$ exist in X. Let, further, $\lambda_n \geqslant 0$ and $\lambda_n \to 0$. Then $\sup \lambda_n x_n \in N$.

Theorem 3 is valid, in particular, also for K-spaces and gives a condition under which the factor X/N turns out to be Archimedean; at the same time the factor nevertheless may fail to be a K'-space (and even a K-space), as is seen from the example given at the end of the note.

Let X be a structure and $H \subset X$. We shall say that H is **structurally σ-closed** if H is closed with respect to the operations sup and inf, i.e. if the following condition is fulfilled in H: let $x_n \in H$ and let $x = \sup x_n$ or $x = \inf x_n$ exist in X; then $x \in H$.

Theorem 4. Let X be an extended K-space, N its normal subspace. Then, in order that the factor X/N be Archimedean, it is necessary and sufficient that N be structurally σ-closed. In this case X/N turns out to be a K'-space.

If X is an extended K-space of countable type, then, in order that X/N be Archimedean, it is necessary and sufficient that N be a component in X.

It is clear that if X is a K-space and N is its component, then the factor X/N is isomorphic to the complementary component and, consequently, will also be a K-space.

Theorem 5. Let X be a K-space with convergence with regulator*; let N be its normal subspace. Then, in order that the factor X/N be Archimedean, it is necessary and sufficient that N be structurally σ-closed.

If X is, moreover, of countable type, then, in order that the factor X/N be Archimedean, it is necessary and sufficient that N be a component in X.

Remark. If one postulates the continuum hypothesis, then a K-space with convergence with regulator turns out to be of countable type ([1], p. 176). Then the theorem can be formulated as follows: if $X-$

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* That is, (o)-convergence in X has the property formulated in ([1], Chap. V, Theorem 1.24).

K-space with convergence with a regulator, and N is its normal subspace, then in order that X/N be Archimedean it is necessary and sufficient that N be a component.

We shall call a sublinear T of a KB-lineal X (b)**-closed** if from $x_n \in T$ and $x_n \xrightarrow{(b)} x \in X$ it follows that $x \in T$.

Theorem 6. *Let X be an Archimedean K-lineal of bounded elements with unit, and let N be its normal sublinear. Then, in order that the quotient X/N be Archimedean, it is necessary and sufficient that N be (b)-closed. Moreover, X/N turns out to be an Archimedean K-lineal of bounded elements with unit.*

Let the K-lineal X under consideration be realized in the form of the space $\mathfrak{C}(Q)$ of certain bounded continuous functions on a bicompact Q (such a realization is possible; see, for example, ([2]), Theorem 1*). In this case Theorem 6 may be given the following form:

In order that X/N be Archimedean, it is necessary and sufficient that N satisfy the following condition:

In N there is the set of all functions from $\mathfrak{C}(Q)$ that vanish on some (obviously, closed) set $Q_0 \subseteq Q$.

Remark. For the K-space of bounded elements, the set of (b)-closed normal subspaces considered in Theorem 6 coincides with the set of (b)-closed ideals ([4]).

In conclusion let us consider one more example. Let $X = m$, i.e. X is the K-space of bounded sequences; $N = c_0$ is the subspace of all sequences tending to zero. Obviously, c_0 is a normal (b)-closed subspace in m; hence, by Theorem 6, m/c_0 is an Archimedean K-lineal. We shall show that the quotient m/c_0 is not even a K-space.

Let x_n be the unit element in m, $x_n \overline{\in} c_0$ $(n = 1, 2, ...)$, and $x_n d x_p$ for $n \neq p$. Suppose that in m/c_0 there exists $\tilde{x} = \sup \tilde{x}_n$. It is clear that $0 < \tilde{x} \leq 1$, where 1 is the image of the unit $1 \in m$ in m/c_0. Take $x \in \tilde{x}$ such that $0 < x \leq 1$, and put $x'_n = x_n \cap x$. It is obvious that $x'_n d x'_p$ $(n \neq p)$, $x'_n \in \tilde{x}_n$ (this follows from the fact that $X \to X/N$ is a lattice homomorphism), and that for every n there is a k_n such that $\xi^{(n)}_{k_n} = \alpha_n > 1/2$ ($\xi^{(n)}_p$ is the p-th coordinate of the element x'_n). Denote by e_p the p-th unit vector in m and put $z_n = \alpha_n e_{k_n}$, and $x''_n = x'_n - z_n$. It is clear that $z_n \in c_0$, $z_n > 0$, $x''_n \in \tilde{x}_n$, and $x''_n d x''_p$ $(n \neq p)$. Let $x'' = \sup x''_n$, $x_0 = \sup z_n$; obviously, $x_0 \overline{\in} c_0$. Now, on the one hand, we have $x'' \geq x_n$, whence $\tilde{x}'' \geq \tilde{x}$. On the other hand, $x'' = \sup x''_n = \sup\{x'_n - z_n\} = \sup x'_n - \sup z_n$ (for $-x'_n > z_n > 0$ and $x'_n d x'_p$ when $n \neq p$), $x'' = \sup x'_n - x_0 \leq x - x_0$, i.e. $\tilde{x}'' \leq \tilde{x} - \tilde{x}_0 < \tilde{x}$. The contradiction obtained proves our assertion.

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* In ([2]) a realization of X is given in the form $\mathfrak{C}(Q)$ on a "minimal" bicompact Q; namely, all points of the bicompact Q are functionally separated by means of $\mathfrak{C}(Q)$, i.e. for any $t_1, t_2 \in Q$ $(t_1 \neq t_2)$ there exists a continuous function $x(t)$ from $\mathfrak{C}(Q)$ for which $x(t_1) \neq x(t_2)$.

Note: Figure translations are in progress. See original paper for figures.

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