Reports of the Academy of Sciences of the USSR

1958

SovietRxiv

Abstract

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Reports of the Academy of Sciences of the USSR
1958. Volume 122, No. 1

THEORY OF ELASTICITY

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SOME QUESTIONS OF THE STABILITY OF SHELLS IN THE LARGE

(Presented by Academician V. I. Smirnov, 12 V 1958)

In solving many problems of stability of elastic systems, the method of linearization is widely used; according to it, the moment of loss of stability is determined by the first eigenvalue of a certain linear boundary-value problem. The question of the legitimacy of this method of solving stability problems for rods was considered in ([1]) and finally resolved in ([2]). At the same time it is known ([3],[4]) that, in the problem of stability of shells, linearization of the equations cannot always be applied. Here we present some general facts concerning this question, obtained on the basis of a rigorous analysis of the fundamental equations of the nonlinear theory of shells.

We shall assume that: 1) the middle surface of the shell Σ is given by the equation $\mathbf{r} = \mathbf{r}(\alpha, \beta)$; $\alpha, \beta \in \bar{\Omega}$; Ω is a bounded domain of the α, β-plane; 2) the boundary $\partial\Omega = \Gamma_\Omega$ consists of a finite number of arcs, on each of which the tangent rotates continuously; 3) $\mathbf{r}$ has continuous derivatives of the second order in $\bar{\Omega}$. Suppose that the external forces acting on the shell have the form $\lambda X, \lambda Y, \lambda Z$, where λ is a numerical parameter; $X, Y, Z \in L_p$, $p > 1$, and for $\lambda = 1$ there exists a momentless stressed state. We shall further assume that the equations of deformation of the shell can be simplified by the method of Kh. M. Mushtari ([5],[6]). In this case a momentless stressed state will exist for any λ, but it will by no means always be unique; and, in solving the stability problem, one must: 1) give a description of the number $n(\lambda)$ of possible stressed states of the shell for different λ; 2) determine the degree of reality of each of the forms of equilibrium of the shell, if the number of forms exceeds unity ($n(\lambda) \geq 1$), as was shown in ([7]).

As an example, let us consider the case of a clamped shell. In this case the first part of the stability problem reduces to the study of the number of solutions $n(\lambda)$ of the nonlinear boundary-value problem (in what follows, for brevity, we shall call it NBVP) for the system:

$$B\left\{\frac{1}{AB}\left[(Bu)_\alpha+(Av)_\beta\right]\right\}_\alpha+\frac{1-\sigma}{2}A\left\{\frac{1}{AB}\left[(Au)_\beta-(Bv)_\alpha\right]\right\}_\beta+\frac{1-\sigma}{R_1R_2}ABu$$

$$=-\left(\frac{Bw}{R_1}+\frac{Bw_\alpha^2}{2A^2}\right)_\alpha-\sigma\left(-\frac{Bw}{R_2}+\frac{w_\beta^2}{2B}\right)_\alpha$$

$$-\left[\frac{A(1-\sigma)}{2}\left(\frac{2w}{R_{12}}+\frac{w_\alpha w_\beta}{AB}\right)\right]_\beta$$

$$-\frac{A_\beta}{2}(1-\sigma)\left(\frac{2w}{R_{12}}+\frac{w_\alpha w_\beta}{AB}\right)+B_\alpha\left[-\frac{w}{R_2}+\frac{w_\beta^2}{2B^2}+\sigma\left(-\frac{w}{R_1}+\frac{w_\alpha^2}{2A^2}\right)\right]; \qquad (1$$

$$A\left\{\frac{1}{AB}\left[(Av)_\beta+(Bu)_\alpha\right]\right\}_\beta+\frac{1-\sigma}{2}B\left\{\frac{1}{AB}\left[(Bv)_\alpha-(Au)_\beta\right]\right\}_\alpha+\frac{1-\sigma}{R_1R_2}ABw$$

$$=-\left(-\frac{Aw}{R_2}+\frac{Aw_\beta^2}{2B^2}\right)_\beta-\sigma\left(-\frac{Aw}{R_1}+\frac{w_\alpha^2}{2A}\right)_\beta-\left[\frac{B(1-\sigma)}{2}\left(\frac{2w}{R_{12}}+\frac{w_\alpha w_\beta}{AB}\right)\right]_\alpha$$

$$-\frac{B_\alpha}{2}(1-\sigma)\left(\frac{2w}{R_{12}}+\frac{w_\alpha w_\beta}{AB}\right)+A_\beta\left[-\frac{w}{R_1}+\frac{w_\alpha^2}{2A^2}+\sigma\left(-\frac{w}{R_2}+\frac{w_\beta^2}{2B^2}\right)\right]; \quad (2)$$

$$AB\nabla^4 w-\frac{1}{E_2}\left\{T_1\left[\frac{AB}{R_1}+\frac{A_\beta w_\beta}{B}+B\left(\frac{w_\alpha}{A}\right)_\alpha\right]+T_2\left[\frac{AB}{R_2}+\frac{B_\alpha w_\alpha}{A}+A\left(\frac{w_\beta}{B}\right)_\beta\right]\right.$$

$$\left.-2S\left(\frac{AB}{R_{12}}+\frac{A_\beta w_\alpha}{A}+\frac{B_\alpha w_\beta}{B}-w_{\alpha\beta}\right)\right\}$$

$$=-\frac{\lambda}{E_2}\left\{T_1'\left[\frac{A_\beta w_\beta}{B^2}+B\left(\frac{w_\alpha}{A}\right)_\alpha\right]+T_2'\left[\frac{B_\alpha w_\alpha}{A^2}+A\left(\frac{w_\beta}{B}\right)_\beta\right]\right.$$

$$\left.-2S'\left(\frac{A_\beta w_\alpha}{A}+\frac{B_\alpha w_\beta}{B}-w_{\alpha\beta}\right)-w_\alpha BX-w_\beta AY\right\}; \qquad (3)$$

$$u\,\big|_{\Gamma_\Omega}=v\,\big|_{\Gamma_\Omega}=0; \qquad (4)$$

$$w\,\big|_{\Gamma_\Omega}=\frac{\partial w}{\partial n}\bigg|_{\Gamma_\Omega}=0. \qquad (5)$$

In (1)–(5), u, v, w are the displacements of points of the middle surface of the shell; A, B are the coefficients of its first form; $\frac{1}{R_1}, \frac{1}{R_2}, \frac{1}{R_{12}}$ are the curvatures of the shell; T_1', T_2', S' are the tangential forces in the momentless stressed state for $\lambda = 1$.

We shall assume that for any w satisfying the boundary conditions (5), the relation

$$I' = \int_\Omega \left(T_1' \frac{w_\alpha^2}{A^2} + T_2' \frac{w_\beta^2}{B^2} + 2S' \frac{w_\alpha w_\beta}{AB} \right) AB\, d\alpha\, d\beta \geq 0$$

holds, and that from $I' = 0$ it follows that $w \equiv 0$. Along with our nonlinear boundary-value problem (NBVP), let us consider a certain linear boundary-value problem (LBVP) for the system obtained from (1)–(5) if it is linearized with respect to w. Such a system would be obtained if we tried to solve the shell-stability problem by Euler' s linearization method.

Lemma. The points of the spectrum of the LBVP form a countable set on the positive axis, having a single limit point at $+\infty$.

The smallest eigenvalue of the LBVP will be denoted below by λ_{E} (Euler' s λ).

Theorem 1. The spectrum of the NBVP is situated to the right of a certain point $\lambda_{\mathrm{n}} < \infty$, and any half-interval $\lambda_{\mathrm{n}} \leq \lambda < \lambda_{\mathrm{n}} + \varepsilon$, where $\varepsilon > 0$ is an arbitrarily small number, contains at least one point of the spectrum. Moreover, the entire half-line $\lambda > \lambda_{\mathrm{e}}$ belongs to the spectrum of the NBVP.

By Theorem 1, $n(\lambda) = 1$ if $\lambda < \lambda_{\mathrm{n}}$.

Theorem 2. There exists a number λ_{y} such that for all $\lambda > \lambda_{\mathrm{y}}$, along with the momentless equilibrium form, the shell has at least one moment equilibrium form with a lower level of the potential energy of the system.

If $\lambda < \lambda_{\mathrm{y}}$, either no moment equilibrium forms of the shell exist, or each moment form has a higher level of the potential energy of the system than the momentless one. In this case the inequalities

$$\lambda_{\mathrm{n}} \leq \lambda_{\mathrm{y}} \leq \lambda_{\mathrm{e}}.$$

Theorem 3. *If $R_1 = R_2 = R_{12} = \infty$ and, consequently, the shell turns into a plate, then $\lambda_{\mathrm{n}} = \lambda_{\mathrm{y}} = \lambda_{\mathrm{e}}$.*

Thus, for a plate, Theorems 1, 2, and 3 completely solve the question of the structure of the lks spectrum. Namely, if $\lambda \leq \lambda_{\mathrm{e}}$, then $n(\lambda) = 1$; if $\lambda > \lambda_{\mathrm{e}}$, then $n(\lambda) \geq 2$.

Theorem 3 may be regarded as a justification of the method of linearization in solving stability problems for plates.

We shall now formulate a certain criterion whose fulfillment makes it impossible to use the linearization method. It is known that, under the assumptions of Kh. M. Mushtari, the potential energy of the system I can be represented in the form

$$I = I^{(2)}(w) + I^{(3)}(w) + I^{(4)}(w),$$

where $I^{(k)}(w)$, $k = 2, 3, 4$, are certain functionals, homogeneous with respect to w, of order k.

Theorem 4. *Suppose there exists an element χ belonging to the proper subspace λ_{e} and such that $I^{(3)}(\chi) \neq 0$. In this case the strict inequality $\lambda_{\text{y}} < \lambda_{\text{e}}$ holds.*

Thus, when the relation $I^{(3)}(\chi) \neq 0$ is satisfied, the lks spectrum continues continuously to the left from λ_{e}. Therefore, in the present case, linearization is impossible in solving the stability problem. The condition $I^{(3)}(\chi) \neq 0$ can be checked for various cases of fixation of cylindrical, conical, and spherical shells. In some of these cases the necessity of passing to a nonlinear theory in solving stability problems had previously been established by means of an approximate solution of the problem by some direct method ([3,4]).

Theorem 5. *In order that the point λ be a bifurcation point of the lks, it is necessary and sufficient that λ be an eigenvalue of the lks.*

The proof of this fact is based on certain properties of functionals close to quadratic ones ([8]).

We shall give some facts characterizing the behavior of shells after loss of stability.

Theorem 6. *To each level of the potential energy I of a plate after loss of stability there corresponds a countable number of eigenfunctions w_n of the lks. Moreover, all the corresponding eigenvalues satisfy $\lambda_n > \lambda_{\text{e}}$, and $\lambda_n \to \infty$ as $n \to \infty$.*

The proof of this fact can be obtained by observing that the study of the behavior of a plate after loss of stability is equivalent to the study of the critical points of a weakly continuous even functional on the unit sphere in the space of bending energy ([8]). In doing so, one may use the theory given in ([9–12]).

Let us say a few words about the second question of the stability problem—about the degree of reality of each equilibrium form. If $\lambda < \lambda_{\text{n}}$, then (Theorem 1) this question does not arise, since in this case the shell possesses a single momentless equilibrium form. For $\lambda > \lambda_{\text{e}}$ it can be shown that the momentless equilibrium form ceases to be a relative minimum of the potential energy of the shell. If, however, $\lambda_{\text{n}} < \lambda < \lambda_{\text{e}}$, then, alongside the momentless form, stable moment forms may also exist. The solution of the question of the degree of reality of each of these forms requires the introduction of probabilistic considerations. Such a probabilistic interpretation of the given problem can be constructed approximately. In doing so, the scattering of the characteristics of the shell itself, of the method of its fixing, and of the external forces is taken into account. It is assumed that the external forces have two components, one of which produces accelerations of points of the shell of the type of accelerations of Brownian motion, while the other is a continuous random process. The determination of the scattering in

the deflections of the shell reduces to solving the Smoluchowski equation and to the subsequent application of the theorem on total probability. In many practically important cases, calculation formulas are obtained directly. On the basis of this interpretation one can, for example, characterize the necessary accuracy in manufacturing the shell (chiefly in producing the shape of the middle surface), which would guarantee that the shell remains, with one probability or another, in a momentless stressed state.

In conclusion, we note that all the facts listed above remain valid for a number of other cases of shell fastening as well.

Received
28 VI 1957

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Note: Figure translations are in progress. See original paper for figures.