Reports of the Academy of Sciences of the USSR

1958

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR

1958. Vol. 120, No. 4

MATHEMATICS

S. I. Zukhovitskii

AN ALGORITHM FOR CONSTRUCTING THE CHEBYSHEV APPROXIMATION OF A CONTINUOUS FUNCTION BY A POLYNOMIAL

(Presented by Academician N. N. Bogolyubov on 13 XII 1957)

1. Suppose that on some compact set Q there is given a real continuous function $f(q)$ and a system of n real continuous and linearly independent functions $\varphi_1(q), \dots, \varphi_n(q)$. By the problem of the Chebyshev approximation of the function $f(q)$ by means of a polynomial of the form $\xi_1\varphi_1(q)+\dots+\xi_n\varphi_n(q)$ one means the problem of finding a system of coefficients $(\xi_1^0, \dots, \xi_n^0) = x^0$ such that the deviation

$$\max_{q\in Q} |\Delta(x^0, q)| = \max_{q\in Q} \left| \sum_{k=1}^{n} \xi_k^0 \varphi_k(q) - f(q) \right|$$

is least.

For some problems of Chebyshev approximation on an interval, algorithms were created by P. L. Chebyshev ([1]) and S. N. Bernstein ([2]) which make it possible to construct a sequence of polynomials whose deviations converge to the least deviation. Some algorithms were indicated by E. Ya. Remez ([3]).

In papers ([4,5]) a finite monotone algorithm was constructed for the Chebyshev approximation of a finite system of inconsistent linear equations. In these same papers it was shown that, in order to construct a polynomial whose deviation from $f(q)$ is approximately least, it is sufficient to consider an η-net $q_1, \dots, q_m$ of the compact set Q and, by this algorithm, to find the Chebyshev approximation of the system of inconsistent linear equations

$$\varphi_1(q_i)\xi_1 + \varphi_2(q_i)\xi_2 + \dots + \varphi_n(q_i)\xi_n = f(q_i) \quad (i = 1, \dots, m), \tag{1}$$

i.e., to find a point $x^* = (\xi_1^*, \dots, \xi_n^*)$ for which

$$\max_i \left| \sum_{k=1}^n \varphi_k(q_i)\xi_k^* - f(q_i) \right| = \inf_x \max_i \left| \sum_{k=1}^n \varphi_k(q_i)\xi_k - f(q_i) \right|,$$

and, for sufficiently small $\eta > 0$, the deviation of the polynomial $\xi_1^*\varphi_1(q) + \dots + \xi_n^*\varphi_n(q)$ from $f(q)$ will differ arbitrarily little from the least one. Using one idea of Vallée-Poussin, it is not difficult to show (see, for example, ([6])) how, in the case of approximation on an interval by an ordinary or trigonometric polynomial, for a given $\varepsilon > 0$ one can actually choose $\eta > 0$ such that the deviation of the approximate polynomial differs by less than ε from the least deviation.

In the present paper we give an algorithm for directly constructing the Chebyshev approximation of a function continuous on a compact set, without a preliminary passage to the system (1), and consequently without preliminary computation of the values of the functions $\varphi_1(q), \dots, \varphi_n(q), f(q)$ on an η-net. In this algorithm, one has mainly to deal only with

by the values of the functions at points of the compact set Q that are especially important for our problem, at which the modulus of the difference between the approximating polynomial and the function $f(q)$ has the greatest value, i.e. at points whose choice is predetermined by the nature of the functions themselves.

2. **For the first approximation** to the point x^0, take an arbitrary point $x_1 = (\xi_1^{(1)}, \dots, \xi_n^{(1)})$ and find on Q the maximum value of the modulus of the function

$$\Delta(x_1, q) = \xi_1^{(1)}\varphi_1(q) + \xi_2^{(1)} \cdot \dots \cdot \varphi_2(q) + \dots + \xi_n^{(1)}\varphi_n(q) - f(q).$$

Let $\max_{q\in Q} |\Delta(x_1, q)|$ be attained only at one point $* \; q_0 \in Q$, so that

$$|\Delta(x_1, q_0)| > |\Delta(x_1, q)|$$

for all $q \in Q - \{q_0\}$.

We now determine the direction of the gradient of the function $|\Delta(x, q_0)|$ at the point x_1, i.e. consider the function of $z = (\zeta_1, \dots, \zeta_n)$ and of the parameter ε

$$|\Delta(x_1 + \varepsilon z, q_0)| = \left| \Delta(x_1, q_0) + \varepsilon \sum_{k=1}^n \varphi_k(q_0)\zeta_k \right|,$$

and define $z = z_1$ so that the derivative

$$\frac{d}{d\varepsilon} \left| \Delta(x_1, q_0) + \varepsilon \sum_{k=1}^n \varphi_k(q_0)\zeta_k \right|_{\varepsilon=0} = \operatorname{sign} \Delta(x_1, q_0) \cdot \sum_{k=1}^n \varphi_k(q_0)\zeta_k$$

is greatest in absolute value and negative for

$$\|z\| = \left[\sum_{k=1}^{n} \zeta_k^2\right]^{1/2} = \text{const.}$$

We obtain

$$z_1 = \{\lambda\varphi_k(q_0)\}_{k=1,\dots,n} = \{\zeta_k^{(1)}\}_{k=1,\dots,n},$$

where λ is an arbitrary multiplier with sign opposite to the sign of $\Delta(x_1, q_0)$.

Now take a sufficiently small number $\eta_1 > 0$ and denote by Q_1 the new compact set obtained by throwing out from Q the entire interior of the sphere of radius η_1 with center at the point q_0, except for the point q_0 itself. Next, from the equation

$$\Delta(x_1 + \varepsilon z_1, q_0) = \delta\Delta(x_1 + \varepsilon z_1, q),$$

where $q \in Q - \{q_0\}$ and $\delta = \pm 1$, define the function

$$\varepsilon(q) = \frac{\delta\Delta(x_1, q) - \Delta(x_1, q_0)}{\sum_{k=1}^{n}[\varphi_k(q_0) - \delta\varphi_k(q)]\zeta_k^{(1)}} \qquad (q \in Q_1 - \{q_0\};\ \delta = \pm 1).$$

Let $\varepsilon(q_1)$ be the smallest positive value of this function on $Q_1 - \{q_0\}$ **.

For the second approximation to the point x^0, we take the point

$$x_2 = x_1 + \varepsilon(q_1)z_1 = \{\xi_k^{(1)} + \varepsilon(q_1)\zeta_k^{(1)}\}_{k=1,\dots,n} = \{\xi_k^{(2)}\}_{k=1,\dots,n}.$$

At this point we have:

$$|\Delta(x_1, q_0)| > |\Delta(x_2, q_0)| = |\Delta(x_2, q_1)| > |\Delta(x_2, q)| \quad (q \in Q_1 - \{q_0, q_1\}),$$

$$\Delta(x_2, q_0) = \delta_1\Delta(x_2, q_1),$$

where δ_1 is equal to $+1$ or -1.

Denote by Q_2 the new compact set obtained by throwing out from Q_1 the entire interior of the sphere of radius η_1 with center at the point q_1, except for the point q_1 itself.

* If this maximum is attained at more than one point, this only accelerates the process.

** If the smallest positive value of the function $\varepsilon(q)$ is attained at more than one point q_1, this only accelerates the process.

Suppose that the p-th approximation x_p $(p \leqslant n)$ has already been constructed so that

$$\left|\Delta(x_p, q_0)\right| = \left|\Delta(x_p, q_1)\right| = \dots = \left|\Delta(x_p, q_{p-1})\right| > \left|\Delta(x_p, q)\right|$$

$$(q \in Q_{p-1} - \{q_0, q_1, \dots, q_{p-1}\}),$$

$$\Delta(x_p, q_0) = \delta_1 \Delta(x_p, q_1) = \dots = \delta_{p-1}\Delta(x_p, q_{p-1}) \quad (\delta_1, \dots, \delta_{p-1} = \pm 1),$$

and let Q_p be the compact set obtained by removing from Q_{p-1} the interior of a sphere of radius η_1 with center at the point q_{p-1}, except for the point q_{p-1} itself.

To construct the $(p+1)$-st approximation, we determine the direction of the relative gradient of the function $|\Delta(x, q_0)|$ at the point x_p, i.e., we determine $z = z_p$ so that the derivative

$$\frac{d}{d\varepsilon}\left|\Delta(x_p + \varepsilon z, q_0)\right|\Big|_{\varepsilon=0} = \operatorname{sign}\Delta(x_p, q_0) \cdot \sum_{k=1}^{n} \varphi_k(q_0)\zeta_k$$

be greatest in absolute value and negative under the conditions:

$$1) \quad \|z\| = \left(\sum_{k=1}^{n} \zeta_k^2\right)^{1/2} = \text{const};$$

$$2) \quad \Delta(x_p + \varepsilon z, q_0) = \delta_1 \Delta(x_p + \varepsilon z, q_1) = \dots = \delta_{p-1}\Delta(x_p + \varepsilon z, q_{p-1}) \quad (\text{i.e.,}$$

$$\sum_{k=1}^{n} \left[\varphi_k(q_0) - \delta_\nu \varphi_k(q_\nu)\right] \zeta_k = 0 \quad (\nu = 1, \dots, p-1)).$$

We obtain

$$z_p = \lambda \left\{\operatorname{sign}\Delta(x_p, q_0) \cdot \varphi_k(q_0) + \right.$$

$$+\sum_{\nu=1}^{p-1}\mu_\nu\left[\varphi_k(q_0)-\delta_\nu\varphi_k(q_\nu)\right]\Bigg\}_{k=1,\dots,n}=\{\zeta_k^{(p)}\}_{k=1,\dots,n},$$

where $\mu_1,\dots,\mu_{p-1}$ are determined from the system

$$\sum_{k=1}^{n}\left[\varphi_k(q_0)-\delta_1\varphi_k(q_1)\right]\left[\varphi_k(q_0)-\delta_i\varphi_k(q_i)\right]\mu_1+$$

$$+\sum_{k=1}^{n}\{\left[\varphi_k(q_0)-\delta_2\varphi_k(q_2)\right]\left[\varphi_k(q_0)-\delta_i\varphi_k(q_i)\right]\}\mu_2\cdots$$

$$\cdots+\sum_{k=1}^{n}\left\{\left[\varphi_k(q_0)-\delta_{p-1}\varphi_k(q_{p-1})\right]\left[\varphi_k(q_0)-\delta_i\varphi_k(q_i)\right]\right\}\mu_{p-1}=$$

$$=-\operatorname{sign}\Delta(x_p,q_0)\cdot\sum_{k=1}^{n}\{\varphi_k(q_0)\left[\varphi_k(q_0)-\delta_i\varphi_k(q_i)\right]\}\quad(i=1,\dots,p-1),$$

and the sign of the arbitrary λ is chosen so that

$$\operatorname{sign}\Delta(x_p,q_0)\cdot\sum_{k=1}^{n}\varphi_k(q_0)\zeta_k^{(p)}<0.$$

Consider the equation

$$\Delta(x_p+\varepsilon z_p,q_0)=\delta\Delta(x_p+\varepsilon z_p,q)\quad(q\in Q_p-\{q_0,\dots,q_{p-1}\};\ \delta=\pm1)$$

and from it find the function

$$\varepsilon(q)=\frac{\delta\Delta(x_p,q)-\Delta(x_p,q_0)}{\sum_{k=1}^{n}\left[\varphi_k(q_0)-\delta\varphi_k(q)\right]\zeta_k^{(p)}}\quad(q\in Q_p-\{q_0,\dots,q_{p-1}\};\ \delta=\pm1).$$

Let $\varepsilon(q_p)$ be the smallest positive value of this function.

As the $(p+1)$-st approximation to the point x^0 we take the point

$$x_{p+1}=x_p+\varepsilon(q_p)z_p=\{\xi_k^{(p)}+\varepsilon(q_p)\zeta_k^{(p)}\}_{k=1,\dots,n}=\{\xi_k^{(p+1)}\}_{k=1,\dots,n},$$

at which

$$|\Delta(x_p, q_0)| > |\Delta(x_{p+1}, q_0)| = |\Delta(x_{p+1}, q_1)| = \cdots$$

$$\cdots = |\Delta(x_{p+1}, q_p)| > |\Delta(x_{p+1}, q)| \quad (q \in Q_p - \{q_0, \dots, q_p\}),$$

$$\Delta(x_{p+1}, q_0) = \delta_1 \Delta(x_{p+1}, q_1) = \cdots = \delta_p \Delta(x_{p+1}, q_p) \quad (\delta_1, \dots, \delta_p = \pm 1).$$

Denote by Q_{p+1} the compact set obtained by removing from Q_p the entire interior of the sphere of radius η_1 with center at the point q_p, except for the point q_p itself.

We shall continue the preceding process until the construction of a compact set Q_{r-1} and a stationary point x_r, i.e., one such that

$$|\Delta(x_r, q_0)| = |\Delta(x_r, q_1)| = \cdots = |\Delta(x_r, q_{r-1})| > |\Delta(x_r, q)|$$

$$(q \in Q_{r-1} - \{q_0, \dots, q_{r-1}\}),$$

but there does not exist such a direction $z = z_r$ along which, under motion (i.e., for increasing $\varepsilon > 0$), all the quantities $|\Delta(x_r + \varepsilon z_r, q_0)|, \dots,$

$$\dots, |\Delta(x_r + \varepsilon z_r, q_{r-1})|$$

would decrease while remaining equal to one another.* If the stationary point x_r lies inside one of the simplexes (or one of the prisms (6)) bounded by the planes of the system

$$\varphi_1(q_i)\xi_1 + \varphi_2(q_i)\xi_2 + \cdots + \varphi_n(q_i)\xi_n = f(q_i) \quad (i = 0, 1, \dots, r-1), \qquad (2)$$

then the polynomial

$$\sum_{k=1}^{n} \xi_k^{(r)} \varphi_k(q)$$

deviates least from $f(q)$ on the compact set Q_{r-1}, and the first part of the process ends here. Otherwise z_r is directed from the point x_r to the vertex of the simplex (edge of the prism) having maximal characteristic, i.e., such that the sum of the number of planes from (2) passing through it and the number of planes from (2) separating it from x_r has the greatest value. Similarly to the preceding steps, we construct the function $\varepsilon(q)$, find its least positive value $\varepsilon(q_r)$, and obtain the next approximation x_{r+1} (i.e., a new stationary point) and a new compact set Q_r.

It is not hard to show that after a finite number of steps the process ends with the construction of a compact set $Q_{m_1} \subset Q$ (possibly consisting only of a finite number of points) and of a point x_1^0 such that the corresponding polynomial deviates least from $f(q)$ on Q_{m_1}, and moreover the inequality

$$\max_{q \in Q_{m_1}} |\Delta(x_1^0, q)| \leq \inf_x \max_{q \in Q} |\Delta(x, q)| \leq \max_{q \in Q} |\Delta(x_1^0, q)|$$

holds, giving a convenient estimate of the accuracy with which the deviation of our polynomial approximates the least deviation on the entire compact set Q.

3. Remark. The process can, of course, be repeated, taking the constructed point x_1^0 as the first approximation and taking as the radius of the spheres to be removed a number $\eta_2 < \eta_1$. As already indicated at the beginning, as $\eta_k \to 0$ the deviation of the corresponding polynomial tends to the least deviation.

Lutsk Pedagogical Institute
named after Lesya Ukrainka

Received
3 XII 1957

CITED LITERATURE

1. P. L. Chebyshev, *Complete Collected Works*, 2, Moscow–Leningrad, 1947.
2. S. N. Bernstein, *Collected Works*, 1, 1952.
3. E. Ya. Remez, *On Methods of the Best, in the Sense of Chebyshev, Approximate Representation of Functions*, Kiev, 1935.
4. S. I. Zukhovitskii, DAN, 79, No. 4 (1951).
5. S. I. Zukhovitskii, *Mathematical Collection*, 33 (75), 2 (1953).

* If the functions $\varphi_1(q), \dots, \varphi_n(q)$ do not form a Chebyshev system on Q, then x_r may turn out to be a stationary point also for $r \leq n$.

Note: Figure translations are in progress. See original paper for figures.