ON EXACT SOLUTIONS OF THE EQUATIONS OF ONE-DIMENSIONAL GAS DYNAMICS WITH SHOCK AND DETONATION WAVES

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Abstract

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ON EXACT SOLUTIONS OF THE EQUATIONS OF ONE-DIMENSIONAL GAS DYNAMICS WITH SHOCK AND DETONATION WAVES

(Presented by Academician L. I. Sedov, 26 IV 1958)

L. I. Sedov found an exact solution (1) of the equations of one-dimensional unsteady motion of an ideal perfect non-heat-conducting gas, in which the velocity v at each instant of time t is a linear function of the distance r from the center, from the axis, or from the plane of symmetry. It is represented by the formulas:

$$dt = \pm \frac{d\mu}{\mu^2 \left[A + B\mu^\nu(\gamma - 1)\right]^{1/2}}, \qquad v = \pm\mu \left[A + B\mu^\nu(\gamma - 1)\right]^{1/2} r,$$

$$\rho = \frac{\mu^{\nu-1}}{r}\varphi'(\mu r), \qquad p = \mu^{\gamma\nu}\left[C + \frac{\nu(\gamma-1)}{2}B\varphi(\mu r)\right], \tag{1}$$

where the density ρ and pressure p are expressed through an arbitrary function $\varphi(\mu r)$ (A, B, and C are arbitrary constants; γ is the ratio of specific heats; $\nu = 1, 2$, and 3 respectively for plane, axial, and central symmetry).

Keller (2), V. P. Korobeinikov and E. V. Ryazanov (5) matched this solution with a shock wave propagating into a gas at rest.

Below a more general method is given for matching solution (1) with a shock or detonation wave propagating with velocity c into a gas at rest with constant initial pressure p_1 and some initial density $\rho_1(r)$. The form of the function $\rho_1(r)$, the function $\varphi(\mu r)$, and the law of motion of the discontinuity surface will be determined from the conditions on the shock or detonation wave.

1. Let the flow behind the shock wave be described by formulas (1). We represent the conditions on the shock wave in the form:

$$\rho_1 = \rho_2 \frac{c - v_2}{c}; \tag{2}$$

$$p_1 = p_2 + \rho_2 v_2^2 - \rho_2 c v_2; \tag{3}$$

$$c = \frac{(\gamma-1)p_1 + (\gamma+1)p_2}{2(\rho_2 - p_1)}\, v_2. \tag{4}$$

Relations (2) and (3) express the laws of conservation of mass and momentum and remain valid also on a detonation wave.

Substituting into (3) from (1) the values of the quantities behind the shock wave and taking into account that

$$c = \frac{dr_2}{dt} = -\frac{v_2\mu}{r_2}\frac{dr_2}{d\mu},$$

we obtain a differential equation for

$\varphi(\mu r_2)$, solving which, we find the pressure behind the shock wave $p_2(\mu)$:

$$p_2(\mu) = p_1 + \frac{\nu\gamma\mu^\nu}{\sqrt{A + B\mu^{\nu(\gamma-1)}}}\left[D + p_1\int_{\mu_0}^{\mu}\frac{\sqrt{A + B\mu^{\nu(\gamma-1)}}}{\mu^{\nu\gamma+1}}\,d\mu\right]. \tag{5}$$

Computing the quadrature in relation (4), we find the radius of the shock wave $r_2(\mu)$:

$$r_2(\mu) = \frac{\delta}{\mu^{\frac{\nu(\gamma+1)}{2}}\left[D + p_1\int_{\mu_0}^{\mu}\frac{\sqrt{A+B\mu^{\nu(\gamma-1)}}}{\mu^{\nu\gamma+1}}\,d\mu\right]}, \tag{6}$$

where D and δ are constants of integration.

If in (1) $B > 0$, then the shock wave emerges from the center: at $t = 0$ ($\mu = \infty$), $r_2 = 0$; moreover, for $\nu = 3$ and $\nu = 2$, $c = \infty$, while for $\nu = 1$, c is finite, and the pressure at the initial instant is infinite. If $B < 0$, then the minimum radius of the shock wave at $t = 0$, $\mu = (-A/B)^{\frac{1}{\nu(\gamma-1)}}$, is different from 0; moreover $c = 0$, and $p_2 = \infty$.

The density behind the shock wave $\rho_2(\mu)$ has the expression

$$\rho_2(\mu) = \frac{2[p_2(\mu) - p_1]^2}{v_2^2(\mu)[(\gamma+1)p_1 + (\gamma-1)p_2(\mu)]}. \tag{7}$$

With the aid of the obtained relations, the density and pressure at any point of the flow behind the shock wave are easily written in parametric form with parameter τ:

$$\rho(\mu, r) = \rho_2(\tau)\left(\frac{\mu}{\tau}\right)^{\nu}, \qquad p(\mu, r) = p_2(\tau)\left(\frac{\mu}{\tau}\right)^{\nu\gamma}, \qquad \mu r = \tau r_2(\tau); \tag{8}$$

$p_2(\tau)$, $r_2(\tau)$, and $\rho_2(\tau)$ are determined by formulas (5), (6), and (7), in which μ is replaced by τ.

The initial density, using the jump relations, is also represented parametrically with parameter ζ:

$$\rho_1(r) = \frac{2[p_2(\zeta) - p_1]^2}{v_2^2(\zeta)[(\gamma-1)p_1 + (\gamma+1)p_2(\zeta)]}, \qquad r = r_2(\zeta); \tag{9}$$

$p_2(\zeta)$ and $r_2(\zeta)$ are determined by formulas (5) and (6), in which μ is replaced by ζ.

For $p_1 = 0$ (^{2}), and also for $A = 0$ (^{3}), the initial density is easily expressed in explicit form.

At the center of symmetry the initial density has the asymptotic expression: for $B > 0$,

$$\rho_1 = \mathrm{const} \cdot r^{-\frac{(3\nu-2)-\gamma(\nu-2)}{\gamma+1}},$$

and for $B = 0$,

$$\rho_1 = \mathrm{const} \cdot r^{-\frac{2\gamma(\nu+1)-2}{\gamma+1}}.$$

It is finite only for $\nu = 3$ and $\gamma \geq 7$. In the case $A > 0$, the initial density must decrease at infinity to zero. For $A < 0$ ($B > 0$), distributions of the initial density are possible that either decrease to zero at infinity, or increase without bound at a finite distance from the center. Flows of the type under consideration with constant initial density are possible only for $p_1 = 0$ and $A = 0$ (the automodel case (1)).

Let us note also that for $A > 0$, $B < 0$, formula (9) determines the initial density for distances from the center not less than the initial radius of the shock wave.

Let $B > 0$. Calculating the total energy of the perturbed motion of the gas described by formulas (1) and (8), we find that it is equal to the sum of the initial thermal energy and the constant energy E_0, released instantaneously at the initial time:

$$\sigma_\nu \int_0^{r_2} \left(\frac{\rho v^2}{2} + \frac{p}{\gamma-1} \right) r^{\nu-1}\, dr = \frac{p_1}{\gamma-1} \frac{\sigma_\nu}{\nu} r_2^\nu + E_0,$$

where

$$E_0 = \frac{\gamma \sigma_\nu \mathcal{E}}{(\gamma-1)\sqrt{B}}, \qquad \sigma_\nu = \frac{1}{2}\,[4\pi(\nu-1) + (\nu-2)(\nu-3)].$$

Therefore the solution obtained may be applied to problems on a point explosion in a medium with an initial density of the type (9) and an initial counterpressure. From the formula given above it follows that for all motions of the class under consideration $E_0 > 0$, and among the initial distributions (9) there are none that would possess dynamical instability of the type described in (4).

For arbitrary B the constructed solution is applicable to problems on pistons expanding according to the law $r = \text{const}/\mu$.

2. Let (1) describe the motion of the gas behind a detonation wave. Relations (2) and (3) at the wave front remain valid, while relation (4) is replaced by the following:

$$c = \frac{(\gamma_1 - 1)[(\gamma_2 - 1)p_1 + (\gamma_2 + 1)p_2]}{2[(\gamma_1 - 1)p_2 - (\gamma_2 - 1)p_1]} v_2 + \frac{Q(\gamma_1 - 1)(\gamma_2 - 1)(p_2 - p_1)}{[(\gamma_1 - 1)p_2 - (\gamma_2 - 1)p_1]v_2}; \quad (10)$$

γ_1 and γ_2 are the ratios of heat capacities of the combustible mixture and the burnt gas; Q is the heat release per unit mass of gas. The pressure behind the detonation front $p_2(\mu)$ is, as before, expressed by formula (5), which followed only from (3) and (1). Substituting into (10) $c = -\dfrac{v_2\mu}{r_2}\dfrac{dr_2}{d\mu}$ and $v_2^2 = \mu^2[A + B\mu^{\nu(\gamma-1)}]r_2^2(\mu)$, we obtain an ordinary linear differential equation with respect to $r_2^2(\mu)$, determining the law of motion of the detonation front. The density behind the detonation wave $\rho_2(\mu)$ is determined through $p_2(\mu)$ and c from (3). The flow behind the wave is described by formulas (8). The initial density

$$\rho_1(r) = \frac{2[p_2(\zeta) - p_1][(\gamma_1 - 1)p_2(\zeta) - (\gamma_2 - 1)p_1]}{(\gamma_1 - 1)[(\gamma_2 - 1)p_1 + (\gamma_2 + 1)p_2(\zeta)]v_2^2(\zeta) + 2Q(\gamma_1 - 1)(\gamma_2 - 1)[p_2(\zeta) - p_1]},$$

$$r = r_2(\zeta).$$

As an example, consider the case $p_1 = 0,\ A = 0,\ \nu = 3,\ \gamma_1 = \gamma_2 = 3$. The initial density has the form $\rho_1 = a[(b^2 + r^2)^{3/2} - b(b^2 + r^2)]^{-1/2}$, where a and b are constants. The detonation wave moves according to the law $r_2^2(t) = 8b\sqrt{Q}\,t + 16Qt^2$. Calculation of the total energy of the burnt gas shows that the flow corresponds to a point explosion in a combustible mixture, in which an energy E_0 is released instantaneously, expressed through a, b, and Q in the form

$$E_0 = \frac{\pi}{3} aQ(8b)^{3/2}.$$

For $\gamma_1 = \gamma_2 = \gamma$, in flows of the type under consideration detonation occurs in the Jouguet regime (the sound speed of the burnt gas relative to the front) only for $\nu = 3,\ p_1 = 0,\ A = 0,\ \gamma = 3$ and for $\nu = 3,\ p_1 \neq 0,\ A \neq 0,\ \gamma = 3$.

Analogous solutions can be constructed for media with an equation of state of the type $p = A(S)(\rho^n - \rho_0^n)$, where S is the entropy.

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Note: Figure translations are in progress. See original paper for figures.