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Abstract

Full Text

MATHEMATICS

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LIE THEOREMS FOR OPERATORS OF GENERALIZED TRANSLATION

(Presented by Academician S. L. Sobolev on 27 VI 1958)

1. In the present note the first and second direct Lie theorems are transferred to operators of generalized translation. The transfer of the first direct Lie theorem to certain special cases of operators of generalized translation has recently been considered by Delsarte ⁽¹⁾.

Let V_n denote a real n -dimensional, sufficiently many times differentiable manifold; let t, s, r, u be points of the space V_n ; $(t_1, \dots, t_n), (s_1, \dots, s_n)$, etc. be the local coordinates of the corresponding points.

A family of operators T^s , defined on some linear space L of functions $f(t), t \in V_n$, is called a family of **operators of generalized translation** if the following conditions ⁽²⁾ are satisfied:

- 1°. The operators T^s are linear.
- 2°. There exists an upper neutral element $s = s_0$, possessing the property that for every $f(t) \in L$

$$T_t^{s_0} f(t) = f(t),$$

i.e. $T^{s_0} = E$, where E is the identity operator.

- 3°. There exists a linear subspace $M \in L$, for all elements $f(t)$ of which s_0 is also a lower neutral element, i.e.

$$T_t^s f(t)|_{t=s_0} = f(s).$$

Obviously, one may assume (and this will be done everywhere below) that, in the local system of coordinates, the point s_0 has coordinates $(0, 0, \dots, 0)$.

- 4°. The operators T^s satisfy the associativity condition

$$T_s^r T_t^s f(t) = T_t^s T_t^r f(t) \tag{1}$$

for every function $f(t) \in L$.

In addition to these conditions, we shall assume that both the function $f(t)$ and the function $u(s, t) = T_t^s f(t)$ are differentiable with respect to all coordinates as many times as will be needed in the course of our computations.

All the main results are stated here without proofs.

2. Definition of infinitesimal operators. The infinitesimal operators of order k for the family of generalized translation operators T^s are the linear operators

$$L_{k_1, \dots, k_n; t}(f) = \left. \frac{\partial^k u}{\partial s_1^{k_1} \dots \partial s_n^{k_n}} \right|_{s=0}, \quad (2)$$

$$\tilde{L}_{k_1, \dots, k_n; s}(f) = \left. \frac{\partial^k u}{\partial t_1^{k_1} \dots \partial t_n^{k_n}} \right|_{t=0}, \quad (3)$$

where $k = k_1 + \dots + k_n$, $u(s, t) = T_t^s f(t)$.

Differentiating the associativity relation (1) k_1 times with respect to s_1 , k_2 times with respect to s_2 , and so on, k_n times with respect to s_n , and then putting $s = 0$, we obtain, on the basis of (2) and (3), that the function $u(r, t) = T_t^r f(t)$ satisfies the system of equations

$$\tilde{L}_{k_1, \dots, k_n; r} u = L_{k_1, \dots, k_n; t} u. \quad (4)$$

System (4) is, for generalized translation operators, an analogue of Lie's first direct theorem.

In the case of Lie groups (in this case the generalized translation T^s is a translation on a group, for example the right translation $T_t^s f(t) = f(t \cdot s)$, where $t \cdot s$ denotes the operation of group multiplication of the elements t and s) one may restrict oneself to infinitesimal operators of the first order

$$L_{\alpha; t}(f) = \left. \frac{\partial u}{\partial s_\alpha} \right|_{s=0}; \quad \tilde{L}_{\alpha; s}(f) = \left. \frac{\partial u}{\partial t_\alpha} \right|_{t=0} \quad (\alpha = 1, 2, \dots, n),$$

which (in the group case) are differential operators of the first order.

It can be shown that, conversely, if all n infinitesimal operators of the first order are linearly independent and are differential operators of the first order, then the manifold V_n is a group (at least in a neighborhood of the neutral element), and the generalized translation T^s is a translation on this group. Thus, if a generalized translation does not reduce to a translation on a group, it is necessary to involve infinitesimal operators of order higher than the first.

Theorem 1. *The following commutation relations hold:*

- 1) $L_{k_1, \dots, k_n; s} T_t^s f(t) = T_t^s L_{k_1, \dots, k_n; t} f(t);$
- 2) $\tilde{L}_{k_1, \dots, k_n; t} T_t^s f(t) = T_t^s \tilde{L}_{k_1, \dots, k_n; t} f(t);$
- 3) $L_{k_1, \dots, k_n; t} \tilde{L}_{j_1, \dots, j_n; t} = \tilde{L}_{j_1, \dots, j_n; t} L_{k_1, \dots, k_n; t}.$

Condition 3) means that the infinitesimal operators L and \tilde{L} (not necessarily of one and the same order) commute.

3. Suppose that the operators T^s can be represented in the form

$$T_t^s f(t) = \int_{V_n} f(u) d_u \sigma(s, t, u), \quad (5)$$

where $\sigma(s, t, E)$ is a function of the points s, t and the set E . Expanding $T_t^s f(t)$ in powers of s and retaining terms of the second order of smallness, we obtain

$$T_t^s f(t) = f(t) + \sum_{\alpha=1}^m s_\alpha L_{\alpha; t}(f) + \sum_{\alpha, \beta=1}^n s_\alpha s_\beta L_{\alpha\beta; t}(f) + o(|s|^2). \quad (6)$$

We now impose the following restrictions on the measure $\sigma(s, t, E)$:

- 1) $\sigma(s, t, V_n) = 1 + \sum_{\alpha=1}^n s_\alpha q_\alpha(t) + \sum_{\alpha, \beta=1}^n s_\alpha s_\beta q_{\alpha\beta}(t) + o(|s|^2);$
- 2) $\int_{V_n} (u_i - t_i) d_u \sigma(s, t, u) = \sum_{\alpha=1}^n s_\alpha b_{\alpha i}(t) + \sum_{\alpha, \beta=1}^n s_\alpha s_\beta c_{\alpha\beta i}(t) + o(|s|^2);$
- 3) $\int_{V_n} (u_i - t_i)(u_j - t_j) d_u \sigma(s, t, u) = \sum_{\alpha, \beta=1}^n s_\alpha s_\beta a_{\alpha\beta, ij}(t) + o(|s|^2),$

where, for at least one pair of indices α, β ,

$$\alpha_{\alpha\beta; ii}(t) \neq 0 \quad (i = 1, 2, \dots, n);$$

$$4) \int_{V_n} |u_i - t_i| |u_j - t_j| |u_k - t_k| d_u \sigma(s, t, u) = o(|s|^2).$$

Expanding $f(u)$ by Taylor's formula in powers of $(u_i - t_i)$ and retaining only terms of second order, we obtain the expansion

$$\begin{aligned}
 T_t^s f(t) &= f(t) \left[1 + \sum_{\alpha=1}^n s_\alpha q_\alpha(t) + \sum_{\alpha,\beta=1}^n s_\alpha s_\beta q_{\alpha\beta}(t) \right] + \\
 &+ \sum_{i=1}^n \frac{\partial f}{\partial t_i} \left[\sum_{\alpha=1}^n s_\alpha b_{\alpha i}(t) + \sum_{\alpha,\beta=1}^n s_\alpha s_\beta c_{\alpha\beta;i}(t) \right] + \\
 &+ \sum_{i,j=1}^n \frac{\partial^2 f}{\partial t_i \partial t_j} \left[\sum_{\alpha,\beta=1}^n s_\alpha s_\beta a_{\alpha\beta;ij}(t) \right] + o(|s|^2) = \\
 &= f(t) + \sum_{\alpha=1}^n s_\alpha \left[\sum_{i=1}^n b_{\alpha i} \frac{\partial f}{\partial t_i} + q_\alpha(t) f \right] + \\
 &+ \sum_{\alpha,\beta=1}^n s_\alpha s_\beta \left[\sum_{i,j=1}^n a_{\alpha\beta;ij}(t) \frac{\partial^2 f}{\partial t_i \partial t_j} + \sum_{i=1}^n c_{\alpha\beta;i}(t) \frac{\partial f}{\partial t_i} + q_{\alpha\beta}(t) f \right] + o(|s|^2).
 \end{aligned} \tag{7}$$

Comparing (6) and (7), we find that the infinitesimal operators of first order $L_{\alpha;t}$ and of second order $L_{\alpha\beta;t}$ have the form

$$L_{\alpha;t}(f) = \sum_{i=1}^n b_{\alpha i}(t) \frac{\partial f}{\partial t_i} + q_\alpha(t) f; \tag{8}$$

$$L_{\alpha\beta;t}(f) = \sum_{i,j=1}^n a_{\alpha\beta;ij}(t) \frac{\partial^2 f}{\partial t_i \partial t_j} + \sum_{i=1}^n c_{\alpha\beta;i}(t) \frac{\partial f}{\partial t_i} + q_{\alpha\beta}(t) f. \tag{9}$$

To determine the form of the infinitesimal operators $\tilde{L}_{\alpha;s}$ and $\tilde{L}_{\alpha\beta;s}$, we require that there exist a measure $\sigma^*(s, t, E)$ such that for any function $f(t) \in M$ (see condition 3°)

$$T_t^s f(t) = \int_{V_n} f(u) d_u \sigma^*(s, t, u),$$

and that the measure σ^* satisfy conditions 1), 2), 3), and 4), in which t_i, t_j, t_k are replaced by s_i, s_j, s_k . Under this assumption, reasoning as before, we obtain

$$\tilde{L}_{\alpha;s}(f) = \sum_{i=1}^n \tilde{b}_{\alpha i}(s) \frac{\partial f}{\partial s_i} + \tilde{q}_\alpha(s) f; \tag{10}$$

$$\tilde{L}_{\alpha\beta;s}(f) = \sum_{i,j=1}^n \tilde{\alpha}_{\alpha\beta;ij}(s) \frac{\partial^2 f}{\partial s_i \partial s_j} + \sum_{i=1}^n \tilde{c}_{\alpha\beta;i}(s) \frac{\partial f}{\partial s_i} + \tilde{q}_{\alpha\beta}(s) f. \tag{11}$$

4. Analogue of Lie' s second direct theorem

In the preceding paragraph we saw that the function of $2n$ variables $u(s, t) = T_t^s f(t)$ must

satisfy the system of equations (we restrict ourselves to operators of the first and second orders)

$$\tilde{L}_{\alpha;s} u = L_{\alpha;t} u; \quad (12)$$

$$\tilde{L}_{\alpha\beta;s} u = L_{\alpha\beta;t} u, \quad (13)$$

where the operators $L_\alpha, \tilde{L}_\alpha, L_{\alpha\beta}, \tilde{L}_{\alpha\beta}$ have the form (8), (9), (10) and (11).

The first-order operators (8) are linearly dependent (otherwise, as we have already noted earlier, the generalized shift coincides with a shift on a group). Therefore the number of distinct equations of the system (12) is less than n .

Some of the operators $L_{\alpha;t}$ may reduce to multiplication by a constant h_α . In this case the corresponding equation of the system (12) turns into an identity, and it should be replaced by the initial condition

$$\left. \frac{\partial u}{\partial s_\alpha} \right|_{s=0} = h_\alpha f(t). \quad (14)$$

We shall consider two cases:

I. The operators

$$\tilde{L}_{\alpha\beta;s}(f) = \tilde{a}_{\alpha\beta}^{ij}(s) \frac{\partial^2}{\partial s_i \partial s_j} + \dots$$

can be solved (algebraically) with respect to all partial derivatives of the second order $\partial^2 f / \partial s_i \partial s_j$. Thus, we must have the largest possible number of linearly independent operators of the second order (namely, $n(n+1)/2$).

II. There are n infinitesimal operators of the second order of the form

$$\tilde{N}_\alpha(f) = \tilde{a}_\alpha^i(s) \frac{\partial^2 f}{\partial s_i^2} + \tilde{a}_\alpha^{ij}(s) \frac{\partial^2 f}{\partial s_i \partial s_j} + \dots \quad (\alpha = 1, 2, \dots, n),$$

and these equations can be solved algebraically with respect to the derivatives $\partial^2 f / \partial s_i^2$ (i.e. the determinant $\det\{\tilde{a}_\alpha^i(s)\}_{\alpha,i=1}^n \neq 0$) and $a_\alpha^{ij}(0) = 0$ for $i \neq j$. To the operators $\tilde{N}_{\alpha;s}$ there correspond the operators $N_{\alpha;t}$, so that a system of the form (13) is satisfied.

From the compatibility condition for the system (12)–(13) one can obtain the following theorems, which should be regarded as a generalization of Lie’s second fundamental theorem.

Theorem 2. In case I there exist constant numbers $\lambda_{i'j'k'l'}^{\alpha\beta\gamma}$, $\mu_{i'j'k'l'}^{\alpha\beta}$, $\nu_{i'j'k'l'}^{\alpha}$, $\tau_{i'j'k'l'}$ such that, if $(ijkl)$ are arbitrary indices and $(i'j'k'l')$ is some permutation of these indices, then

$$(L_{ij}L_{kl} - L_{i'j'}L_{k'l'})f = \lambda_{i'j'k'l'}^{\alpha\beta\gamma}L_{\alpha\beta}L_{\gamma}(f) + \mu_{i'j'k'l'}^{\alpha\beta}L_{\alpha\beta}(f) + \nu_{i'j'k'l'}^{\alpha}L_{\alpha}(f) + \tau_{i'j'k'l'}f. \quad (15)$$

If $L_{\alpha;t}(f) = h_{\alpha}f(t)$ ($\alpha = 1, \dots, n$) and $L_{\alpha\beta;t}(1) = 0$, then identity (15) is simplified and takes the form

$$(L_{ij}L_{kl} - L_{i'j'}L_{k'l'})f = \mu_{i'j'k'l'}^{\alpha\beta}L_{\alpha\beta}(f).$$

Theorem 3. In case II

$$(N_{iN}k - N_{kN}i)f = \lambda_{ik}^{\alpha\beta}N_{\alpha}L_{\beta}(f) + \mu_{ik}^{\alpha}N_{\alpha}(f) + \nu_{ik}^{\alpha}L_{\alpha}(f) + \tau_{ik}f, \quad (16)$$

where λ, μ, ν, τ are constants. If all first-order operators reduce to the operator of multiplication by a constant and $N_{\alpha}(1) = 0$ for all α , then equality (16) is simplified and takes the form

$$(N_{iN}k - N_{kN}i)f = c_{ik}^{\alpha}N_{\alpha}(f),$$

where c_{ik}^{α} are the structure constants of a certain Lie group.

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CITED LITERATURE

¹ J. Delsarte, Hypergroupes et opérateurs de permutation et de transmutation, Colloques internationaux, Nancy, 1956, p. 29. ² B. M. Levitan, *Uspekhi Mat. Nauk*, 4, no. 1, 3 (1949).

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