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Abstract

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MATHEMATICS

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ASYMPTOTIC EXPANSION OF SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS OF SECOND ORDER IN BANACH SPACES

(Presented by Academician S. L. Sobolev, 24 IV 1958)

Statement of the problem: to investigate the behavior of solutions of differential equations of the 2nd order in Banach spaces with a small parameter $\varepsilon > 0$ at the highest derivative y'', when $\varepsilon \to 0$.

1. Let Y be a Banach space with multiplication of elements by real numbers. Let the function $f(\varepsilon) \in Y$ (ε real) be given in some neighborhood of the point $\varepsilon = 0$. Let the series

$$C_0 + C_1\varepsilon + C_2\varepsilon^2 + \cdots + C_n\varepsilon^n + \cdots, \tag{1}$$

where $C_n \in Y$, be such that for any fixed n

$$\lim_{\varepsilon\to 0} \frac{\left\|f(\varepsilon) - (C_0 + C_1\varepsilon + C_2\varepsilon^2 + \cdots + C_n\varepsilon^n)\right\|}{\varepsilon^n} = 0. \tag{2}$$

Then we shall say that (1) serves as an asymptotic expansion of $f(\varepsilon)$, and write this as follows:

$$f(\varepsilon) \sim \sum_{n=0}^{\infty} C_n\varepsilon^n. \tag{3}$$

For an arbitrary choice of elements $C_0, C_1, \dots, C_n, \dots \in Y$, one can construct a function $f(\varepsilon) \in Y$ for which (3) is satisfied.

Consider the equation

$$\varepsilon y'' + y' + Ay = \theta, \tag{4}$$

where $\varepsilon > 0$ is a small parameter; $y(x,\varepsilon) \in Y$; A is any linear bounded operator mapping the space Y into Y, i.e. $A \in \{Y \to Y\}$. We shall consider solutions of equation (4) on an arbitrary fixed interval $[x_0, x_1]$ $(x_0 < x_1)$ of the OX axis. Multiplying (4) by ε and introducing the notation $\varepsilon y' = y^{[1]}$, $\varepsilon^2 y'' = y^{[2]}$, we write the equation in the form

$$A(y) = y^{[2]} + y^{[1]} + \varepsilon Ay = \theta. \tag{5}$$

The set of all possible polynomials of the form $a_0 + a_1A + a_2A^2 + \dots + a_{sA}^s$, where $a_0, a_1, \dots, a_s$ are arbitrary real numbers and s is any natural number, will be denoted by $\{A^s\}$. By virtue of the completeness of the space Y, we can close the space $\{A^s\}$ and obtain a commutative Banach algebra $\overline{\{A^s\}}$. Along with equation (5), consider

$$A(\overline{y}) = \overline{y}^{[2]} + \overline{y}^{[1]} + \varepsilon A\overline{y} = \overline{\theta} \tag{6}$$

in the space $\overline{\{A^s\}}$.

Lemma 1. There exist two infinite sequences of functions
$\bar{u}_{10}(x), \bar{u}_{11}(x), \bar{u}_{12}(x), \dots \in \{\bar{A}^s\}$;
$\bar{u}_{20}(x), \bar{u}_{21}(x), \bar{u}_{22}(x), \dots \in \{\bar{A}^s\}$ for $x \in [x_0, x_1]$, continuous and bounded together with derivatives of all orders; moreover there exist

$$[\bar{u}_{10}]^{-1} = e^{-A(x-x_0)}, \qquad [\bar{u}_{20}]^{-1} = e^{A(x-x_0)}$$

such that: 1) upon substituting into $A(\bar{y})$ in place of $\bar{y}$ the expression

$$\bar{u}_1(x,\varepsilon) = e^{-(x-x_0)/\varepsilon} \sum_{j=0}^{m-1} \bar{u}_{1j}(x)\varepsilon^j$$

in $A(\bar{u}_1(x,\varepsilon))$, the functions multiplying $e^{-(x-x_0)/\varepsilon}\varepsilon^s$ $(s = 0, 1, 2, \dots, m)$ are identically equal to $\bar{\theta}$; 2) upon substituting into $A(\bar{y})$ the expression

$$\bar{u}_2(x,\varepsilon) = \sum_{j=0}^{m-1} \bar{u}_{2j}(x)\varepsilon^j$$

in $A(\bar{u}_2(x,\varepsilon))$, the functions multiplying ε^s $(s = 0, 1, \dots, m)$ are identically equal to $\bar{\theta}$.

Lemma 2. A homogeneous linear differential equation of the 2nd order in $\{\bar{A}^s\}$ with solutions $\bar{u}_1(x,\varepsilon)$, $\bar{u}_2(x,\varepsilon)$ has the form

$$B(\bar{y}) = \bar{y}^{[2]} + b_1(x,\varepsilon)\bar{y}^{[1]} + b_0(x,\varepsilon)\bar{y} = \bar{\theta}. \tag{7}$$

The functions $b_0(x,\varepsilon)$, $b_1(x,\varepsilon) \in \{\bar{A}^s\}$ for $x \in [x_0, x_1]$, $0 \le \varepsilon \le \varepsilon_1$ ($\varepsilon_1 > 0$ sufficiently small) are continuous, uniformly bounded together with their derivatives of all orders with respect to x, and are representable in the form

$$b_0(x,\varepsilon) = \sum_{j=0}^{\infty} b_{0j}(x)\varepsilon^j, \qquad b_1(x,\varepsilon) = \sum_{j=0}^{\infty} b_{1j}(x)\varepsilon^j, \tag{8}$$

where

$$\|b_1(x,\varepsilon) - I\| \le D\varepsilon^{m+1}, \qquad \|b_0(x,\varepsilon) - \varepsilon A\| \le D\varepsilon^{m+1}, \tag{9}$$

$x \in [x_0, x_1]$, $0 \le \varepsilon \le \varepsilon_1$ ($D = \text{const}$, I is the identity operator).

Theorem 1. There exist two fundamental solutions of equation (6), $\bar{y}_1(x,\varepsilon)$, $\bar{y}_2(x,\varepsilon)$, such that

$$\bar{y}_1(x,\varepsilon) = \bar{u}_1(x,\varepsilon) + e^{-(x-x_0)/\varepsilon}\bar{E}_{10}(x,\varepsilon)\varepsilon^m,$$

$$\bar{y}_2(x,\varepsilon) = \bar{u}_1(x,\varepsilon) + \bar{E}_{20}(x,\varepsilon)\varepsilon^m, \qquad \frac{d}{dx}(\bar{y}_1(x,\varepsilon)) = \frac{d}{dx}(\bar{u}_1(x,\varepsilon)) + e^{-(x-x_0)/\varepsilon}\bar{E}_{11}(x,\varepsilon)\varepsilon^{m-1},$$

$$\frac{d}{dx}(\bar{y}_2(x,\varepsilon)) = \frac{d}{dx}(\bar{u}_2(x,\varepsilon)) + \bar{E}_{21}(x,\varepsilon)\varepsilon^{m-1},$$

where the functions $\bar{E} \in \{\bar{A}^s\}$ are continuous in x, with derivatives of all orders on $[x_0, x_1]$, analytic with respect to ε for $0 \le \varepsilon \le \varepsilon^*$ ($0 < \varepsilon^* \le \varepsilon_1$), and uniformly bounded.

In the proof, writing equation (6) in the form

$$\bar{y}^{[2]} + b_1(x,\varepsilon)\bar{y}^{[1]} + b_0(x,\varepsilon)\bar{y} = B(\bar{y}) - A(\bar{y}),$$

one may write the equivalent equality in integral form

$$\bar{y}(x,\varepsilon) = \bar{u}_1(x,\varepsilon)\bar{C}_1 + \bar{u}_2(x,\varepsilon)\bar{C}_2 + \frac{1}{\varepsilon}\int_{x_0}^{x} [\bar{u}_1(x,\varepsilon)\tilde{u}_1(t,\varepsilon) + \bar{u}_2(x,\varepsilon)\tilde{u}_2(t,\varepsilon)][(b_1 - I)\bar{y}^{[1]} + (b_0 - \varepsilon A)\bar{y}]\,dt,$$

where $\bar{C}_1, \bar{C}_2$ are arbitrary constant elements of $\{\bar{A}^s\}$; the functions $\tilde{u}_1(x,\varepsilon)$, $\tilde{u}_2(x,\varepsilon)$ are determined from the equations

$$\bar{u}_1(x,\varepsilon)\tilde{u}_1(x,\varepsilon) + \bar{u}_2(x,\varepsilon)\tilde{u}_2(x,\varepsilon) = \bar{\theta},$$

$$\bar{u}_1^{[1]}(x,\varepsilon)\tilde{u}_1(x,\varepsilon) + \bar{u}_2^{[1]}(x,\varepsilon)\tilde{u}_2(x,\varepsilon) = I;$$

$\bar{y}_1(x,\varepsilon)$, $\bar{y}_2(x,\varepsilon)$ are fundamental in the sense that the operator

$$\Delta = \bar{y}_1\bar{y}_2^{[1]} - \bar{y}_1^{[1]}\bar{y}_2$$

has an inverse for $0 \le \varepsilon \le \varepsilon^*$.

Thus, for $0 \le \varepsilon \le \varepsilon^*$, any solution of equation (4) can be written in the form

$$y(x,\varepsilon) = \bar{y}_1(x,\varepsilon)C_1 + \bar{y}_2(x,\varepsilon)C_2, \tag{10}$$

where C_1, C_2 are arbitrary constant elements of Y. The solution $y(x,\varepsilon)$ of equation (4) is determined by the initial data $y(x_0,\varepsilon) = y_0$, $y'(x_0,\varepsilon) = y_0'$, whereas the solution $y_0(x)$ of the limiting equation is determined only by the value $y_0(x)$ at $x = x_0$, i.e. by y_0. Without loss of generality, we shall assume that $y_0 = \theta$ and, consequently, $y_0(x) \equiv \theta$. For C_1, C_2 in (10) we have: $C_1 = \varepsilon C_1^0(\varepsilon)$, $C_2 = \varepsilon C_2^0(\varepsilon)$, where $C_1^0(\varepsilon), C_2^0(\varepsilon)$ are bounded functions for $0 \leqslant \varepsilon \leqslant \varepsilon^*$.

Thus, $y(x,\varepsilon) \to \theta$ as $\varepsilon \to 0$ for $x \in [x_0, x_1]$; $y'(x,\varepsilon) \to \theta$ as $\varepsilon \to 0$ for $x \neq x_0$; $y''(x,\varepsilon) \to \theta$ as $\varepsilon \to 0$ for $x \neq x_0$, and $\|y''(x_0,\varepsilon)\| \to \infty$ as $\varepsilon \to 0$, of order $1/\varepsilon$, etc. For the solution $y(x,\varepsilon)$ passing through the points (x_0, y_0), (x_1, θ) ($y_0(x) \equiv \theta$ is the solution of the limiting equation passing through the point (x_1, θ)), we have $y(x,\varepsilon) \sim \theta$ for $x \in [x_0, x_1]$, $x \neq x_0$; $x = x_0$ is a boundary-layer point for the solution $y(x,\varepsilon)$ and its derivatives.

2. Consider the equation

$$\varepsilon y'' + p^*(x,\varepsilon)y' + q^*(x,\varepsilon)y = \theta, \tag{11}$$

where the functions p^*, q^*, with values in $\{Y \to Y\}$, are defined for $x \in [x_0, x_1]$, $0 \leqslant \varepsilon \leqslant \tilde{\varepsilon}^*$, are continuous in x, uniformly bounded, and have the form $p^*(x,\varepsilon) = p_0(x) + \varepsilon p_1(x) + \varepsilon^2 p(x,\varepsilon)$, $q^*(x,\varepsilon) = A + \varepsilon q(x,\varepsilon)$. Here $A, p(x,\varepsilon), q(x,\varepsilon) \in \{Y \to Y\}$; $p_1(x) \in \{A^s\}$; $0 < \alpha \leqslant p_0(x)$ is a real function, and $p_0(x)$ and $p_1(x)$ have continuous derivatives of all orders, or at least up to and including order 2.

Theorem 2. There exist two independent solutions of equation (11),

$$y_1(x,\varepsilon),\ y_2(x,\varepsilon) \quad \text{such that} \quad y_1(x,\varepsilon) = \bar{y}_1(x,\varepsilon)C_1 + \exp\left[-\frac{1}{\varepsilon}\int_{x_0}^{x} p_0(t)\,dt\right] \times$$

$$\times E_{10}^{C_1}(x,\varepsilon)\varepsilon, \quad y_1^{[1]}(x,\varepsilon) = \bar{y}_1^{[1]}(x,\varepsilon)C_1 + \exp\left[-\frac{1}{\varepsilon}\int_{x_0}^{x} p_0(t)\,dt\right] E_{11}^{C_1}(x,\varepsilon)\varepsilon, \quad y_2(x,\varepsilon) =$$

$$= \bar{y}_2(x,\varepsilon)C_2 + E_{20}^{C_2}(x,\varepsilon)\varepsilon, \quad y_2^{[1]}(x,\varepsilon) = \bar{y}_2^{[1]}(x,\varepsilon)C_2 + E_{21}^{C_2}(x,\varepsilon)\varepsilon \quad \text{for all}$$

$$x \in [x_0, x_1], \quad 0 \leqslant \varepsilon \leqslant \varepsilon^{**}\ (0 < \varepsilon^{**} \leqslant \tilde{\varepsilon}^*),$$

where C_1, C_2 are arbitrary unit elements of Y; E are functions with values in Y, continuous and uniformly bounded together with their first derivatives with respect to x.

By linear independence is meant the following: any solution of equation (11) can be obtained by a linear combination of the solutions y_1, y_2 with real positive coefficients and with a corresponding choice of the unit elements C_1, C_2, i.e. in the form: $y(x,\varepsilon) = c_1 y_1(x,\varepsilon) + c_2 y_2(x,\varepsilon)$.

Without loss of generality, put $y_0 = \theta$ $(y_0(x) \equiv \theta)$. Then $c_1 = \varepsilon\tilde{c}_1(\varepsilon)$, $c_2 = \varepsilon\tilde{c}_2(\varepsilon)$, where $\tilde{c}_1(\varepsilon), \tilde{c}_2(\varepsilon)$ are bounded for $0 \leqslant \varepsilon \leqslant \varepsilon^{**}$. We obtain: $y(x,\varepsilon) \to \theta$ as $\varepsilon \to 0$, $y'(x,\varepsilon) \to \theta$ as $\varepsilon \to 0$ and $x \neq x_0$.

3. Consider the nonlinear equation

$$\varepsilon y'' = F(x, y, y', \varepsilon), \tag{12}$$

where $F \in Y$ and, in the most general case, has the form:

$$F(x,y,y',\varepsilon) = p_0(x)y' + \varepsilon p_1(x)y' + Ay + \varepsilon a(x,\varepsilon) + \varepsilon b(x,y,y^{[1]},\varepsilon)y + \varepsilon c(x,y,y^{[1]},\varepsilon)y^{[1]} + d(x,y,y^{[1]},\varepsilon)y^2 + e(x,y,y^{[1]}$$

Here real-

the function $p_0(x) \leq \alpha < 0$ and the function $p_1(x) \in \{A^S\}$ are defined for $x \in [x_0, x_1]$ $(x_1 > x_0)$, and are continuous together with their derivatives with respect to the endpoints up to order 2 inclusive; $A \in \{Y \to Y\}$; a, b, c, d, e, f are functions with values respectively in Y, $\{Y \to Y\}$, $\{Y \to Y\}$, $[Y_2 \to Y]$, $[Y_2 \to Y]$, $[Y_2 \to Y]$ ($[Y_2 \to Y]$ is the space of bilinear, symmetric operators), defined for $x_0 \leq x \leq x_1$, $0 \leq \varepsilon \leq \varepsilon_1$, y and $y^{[1]}$ from certain neighborhoods, respectively, of the points $y_0 = \theta \in Y$, $y_0^{(1)} = \theta \in Y$.

The limiting differential equation

$$p_0(x)y' + Ay - d(x, y, \theta, 0)y^2 = \theta \tag{13}$$

has the solution $y_0(x) \equiv \theta$, satisfying $y_0(x_0) = \theta$.

Suppose:

I. The functions a, b, c, d, e, f are continuous in x, uniformly bounded, and analytic with respect to $y, y^{[1]}$, respectively from certain neighborhoods of the points $y_0 = \theta$, $y_0^{[1]} = \theta$.

It follows from I that, for any prescribed element $y_0' \in Y$, equation (12), for $0 \leq \varepsilon \leq \varepsilon_2$ $(0 < \varepsilon_2 < \varepsilon_1)$, has a unique solution $y(x,\varepsilon)$ for $x_0 \leq x \leq r$ $(0 < r < x_1)$, satisfying the initial conditions $y(x_0,\varepsilon) = \theta$, $y'(x_0,\varepsilon) = y'_{0[2]}$.

Theorem 3. The solution $y(x,\varepsilon)$ of equation (12), as $\varepsilon \to 0$, tends to the solution $y_0(x) \equiv \theta$ of the limiting equation (13) to order ε in norm for all $x_0 \leq x \leq x_1$. At the same time, $\|y'(x,\varepsilon)\|$ need not tend to zero as $\varepsilon \to 0$, but remains bounded for all $x_0 \leq x \leq x_1$. An equation that can serve as a realization in the space $C_{[a,b]}$ is

$$\varepsilon \frac{\partial^2 y(x,s)}{\partial x^2} = p_0(x)\frac{\partial y(x,s)}{\partial x} + \varepsilon p_1(x)\int_a^b A(s,t)\int_a^b A(t,\tau)\frac{\partial y(x,\tau)}{\partial x}\,d\tau\,dt +$$

$$+\int_a^b A(s,t)y(x,t)\,dt + \int_a^b\int_a^b\left[\sum_{k=0}^{\infty}\int_a^b\cdots\int_a^b d_k(x,s,\varepsilon,t_1,t_2,\tau_1,\tau_2,\dots,\tau_k)\times\right.$$

$$\left.\times y(\tau_1)y(\tau_2)\cdots y(\tau_k)\,d\tau_1\cdots d\tau_k\right] y(t_1)y(t_2)\,dt_1\,dt_2,$$

where the functions $p_0(x) \le \alpha < 0$, $p_1(x) \in C^{(2)}_{[x_0,x_1]}$; d_{kl} are continuous with respect to all arguments and satisfy the conditions

$$\max_{\substack{x_0\le x\le x_1\\ 0\le\varepsilon\le\varepsilon_1\\ a\le s,t_1,t_2\le b}} \int_a^b\cdots\int_a^b |d_{kl}(x,s,\varepsilon,t_1,t_2,\tau_1,\dots,\tau_n)|\,d\tau_1\,d\tau_2\cdots d\tau_n \le LM^kN^l$$

(L, M, N are constants).

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[1] L. A. Lyusternik, V. I. Sobolev, *Elements of Functional Analysis*, Moscow-Leningrad, 1951.
[2] M. K. Gavurin, *Uchen. zap. LGU*, issue 19, 59 (1950).
[3] E. Hille, *Functional Analysis and Semigroups*, translated from English, Moscow, 1951.
[4] G. Birkhoff, *Trans. Am. Math. Soc.*, 9, 219 (1908).
[5] W. Wasow, *Comm. Math.*, 9, 1, 93 (1956).
[6] J. Ritt, *Ann. Math.*, 18, 18 (1916-1917).

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