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**Abstract**

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MATHEMATICS

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# ON THE RIGIDITY OF AN OPEN SURFACE OF NONNEGATIVE CURVATURE UNDER A NONORTHOGONAL SLEEVE CONSTRAINT

*(Presented by Academician S. L. Sobolev, 11 III 1958)*

Consider an ovaloid $S$ on which there is a finite number of holes. Suppose that into these holes there are inserted certain rigid immovable bodies that fill them tightly. We shall call these bodies **sleeves** [1]. Denote the surfaces of these bodies by $\Sigma$. Thus the surface $S$ is in contact with the surfaces $\Sigma$ along the entire contour. Assume that, in the process of deformation of the surface $S$, the contact between $S$ and $\Sigma$ is not broken. Denoting by $\mathbf{U}$ the displacement vector of points of the ovaloid $S$, the condition of continuous contact between $S$ and $\Sigma$ can be written in the form

$$\mathbf{U}\vec{\nu} = 0 \quad (\text{on } L), \tag{1}$$

where $\vec{\nu}$ denotes the normal to the surface $\Sigma$; $L$ is the collection of the contours of the holes of the surface $S$. Conditions of this kind will be called **sleeve constraints** [1].

In the work [1] I. N. Vekua considered the case of an orthogonal constraint (i.e. $\vec{\nu} \equiv \mathbf{l}$; $\mathbf{l}$ is the tangential normal to $L$), while the case of a nonorthogonal constraint was considered by the author in [2] for surfaces of revolution. In the present work the author considers this problem for a surface of nonnegative curvature ($K \geq 0$) with contours that are circles, the flattening points on the surface being isolated.

1. Denote by $\mathbf{r}$ the radius vector of points of the surface $S$; by $\mathbf{U}$, the bending vector of the surface; and by $\mathbf{V}$, the rotation vector of an infinitesimal bending, which is uniquely determined by the equation

$$d\mathbf{U} = \mathbf{V} \times d\mathbf{r}. \tag{2}$$

Let $L$ be a smooth curve lying on $S$. Let $k$ and $\varkappa$ be its curvature and torsion. At each point of $L$ we may consider the trihedron $\mathbf{m}, \mathbf{b}, \mathbf{s}$, where $\mathbf{s}$ is the unit tangent to $L$; $\mathbf{m}$ and $\mathbf{b}$ are the unit vectors of the principal normal and binormal of $L$, respectively, with

$$\mathbf{b} = \mathbf{s} \times \mathbf{m}, \qquad \frac{d\mathbf{s}}{ds} = k\mathbf{m}, \qquad \frac{d\mathbf{m}}{ds} = -k\mathbf{s} + \varkappa\mathbf{b}, \qquad \frac{d\mathbf{b}}{ds} = -\varkappa\mathbf{m}. \tag{3}$$

Denote by $\mathbf{n}$ the exterior normal to $S$; $\mathbf{n} = \mathbf{m}\cos\theta + \mathbf{b}\sin\theta$, where $\theta$ is the angle between $\mathbf{m}$ and $\mathbf{n}$, measured from $\mathbf{m}$ in the clockwise direction.

We decompose the vectors $\mathbf{U}, \mathbf{V}$ with respect to the trihedron $\mathbf{s}, \mathbf{m}, \mathbf{b}$ in the form

$$\mathbf{U} = u_s\mathbf{s} + u_m\mathbf{m} + u_b\mathbf{b}, \qquad \mathbf{V} = v_s\mathbf{s} + v_m\mathbf{m} + v_b\mathbf{b}. \tag{4}$$

Taking into account that $\frac{dV}{ds} \cdot \mathbf{n} = 0$, the equation of an infinitesimal bending (2) can be reduced to the system

$$v_b = \frac{du_m}{ds} + ku_s - \chi u_b,$$

$$v_m = -\frac{du_b}{ds} - \chi u_m,$$

$$v_s = -\frac{1}{k}\left(\frac{dv_m}{ds} + \frac{du_b}{ds}\tan\theta\right) + \frac{\chi}{k}(v_b - v_m\tan\theta), \tag{5}$$

$$\frac{du_s}{ds} - ku_m = 0.$$

Let $\vec{\nu}$ (the normal to the sleeve) be expressed in the basis $s, \mathbf{m}, \mathbf{b}$ in the form

$$\vec{\nu} = \mathbf{b}\cos\tau - \mathbf{m}\sin\tau, \tag{6}$$

where $\tau$ is the angle between $\vec{\nu}$ and $\mathbf{b}$, and we regard it as positive if it is measured from $\mathbf{b}$ in the clockwise direction. Then the boundary condition (1) can be written in the form

$$u_b\cos\tau - u_m\sin\tau = 0 \quad \text{on } L. \tag{7}$$

In the present work we shall always assume $\tau$ to be constant along each contour of the surface $S$.

It is known that Blaschke' s formula ([3,5]) has the form

$$2\iint_S (\beta\gamma - \alpha^2)(\mathbf{nr})\, dS = \int_L \mathbf{r}V\, dV. \tag{8}$$

Since $\beta\gamma - \alpha^2 \le 0$ for $K \ge 0$ and $(\mathbf{nr}) > 0$, when the origin lies inside the surface, the rigidity of the surface $S$ is an obvious consequence of the inequality

$$\int_L \mathbf{r}V\,dV \ge 0$$

under the boundary condition (7) ([5]).

**Lemma 1** ([4]). If $y(\varphi)$ is a periodic function with period $2\pi$, $y' \in L_2$, and

$$\int_0^{2\pi} y\,d\varphi = 0,$$

then

$$\int_0^{2\pi} y^2\,d\varphi \le \int_0^{2\pi} {y'}^2\,d\varphi.$$

**Lemma 2.** If $L$ is a circle, then the inequalities

$$\int_L k u_m^2\,ds \le \int_L \frac{1}{k}\left(\frac{du_m}{ds}\right)^2 ds, \tag{9}$$

$$\int_L k v_m^2\,ds \le \int_L \frac{1}{k}\left(\frac{dv_m}{ds}\right)^2 ds \tag{10}$$

hold.

**Proof.** 1) Let $y = u_m$. In view of the fact that $k = \dfrac{d\varphi}{ds}$ and

$$\int_0^{2\pi} y\,d\varphi = \int_0^{2\pi} u_m\,d\varphi = \int_L k u_m\,ds = \int_L \frac{du_s}{ds}\,ds = 0,$$

(9) follows immediately from Lemma 1.

2) Since $\chi = 0$ and

$$\int_0^{2\pi} v_m\,d\varphi = \int_0^{2\pi}\left(-\frac{du_b}{ds}\right)d\varphi = -k\int_L du_b = 0,$$

inequality (10) follows immediately from Lemma 1.

2. In this section we shall consider a truncated ovaloid whose contours are circles, i.e., $\varkappa = 0$, $k = \text{const}$ on $L$.

First consider an ovaloid $S$ with one hole. We choose the origin of coordinates so that it lies on the axis of rotation of the circle $L$. Denote by $\omega$ the angle between this axis and the vector $\mathbf{r}$; clearly, $\omega = \text{const}$, if $\mathbf{r}$ runs along the contour $L$.

**Theorem 1.** *The surface $S$ will be rigid if*

$$\max_L \theta - \pi \leqslant \tau \leqslant \pi - \operatorname{arc\,tg}(\max_L \operatorname{tg}\theta + 2\operatorname{tg}\omega_0), \tag{11}$$

*where $\omega_0 = \min \omega$ (with respect to the choice of the origin of coordinates).*

**Proof.** Taking into account that $(\mathbf{rs}) = 0$, $(\mathbf{rb}) = -|\mathbf{r}|\cos\omega$, $(\mathbf{rm}) = -|\mathbf{r}|\sin\omega$, under the boundary condition (7) for system (5) we have

$$\int_L \mathbf{rV}\,dy = |\mathbf{r}| \int_L \left\{ \frac{1}{k}\left(\frac{d^2u_m}{ds^2} + k^2u_m\right)^2 \left[(\operatorname{tg}^2\tau - \operatorname{tg}^2\theta)\cos\omega + 2(\operatorname{tg}\tau - \operatorname{tg}\theta)\sin\omega\right]\right.$$

$$\left. +k^2\left[\frac{1}{k}\left(\frac{du_m}{ds}\right)^2 - ku_m^2\right](\operatorname{tg}^2\tau\cos\omega + \operatorname{tg}\tau\sin\omega)\right\} ds. \tag{12}$$

In order that the integral (12) be nonnegative, it is sufficient, taking Lemma 2 into account, that the angle $\tau$ simultaneously satisfy the following two inequalities:

$$(\operatorname{tg}\tau - \operatorname{tg}\theta)\left[(\operatorname{tg}\tau + \operatorname{tg}\theta)\cos\omega + 2\sin\omega\right] \geqslant 0,$$

$$\operatorname{tg}\tau(\operatorname{tg}\tau\cos\omega + \sin\omega) \geqslant 0. \tag{13}$$

Our theorem follows directly from this.

We note that in order to obtain the result $\max\theta - \pi \leqslant \tau$ in the case $0 < \theta < \pi$, we must choose the origin of coordinates as follows: $O$ lies outside the surface and makes with the axis of rotation of the circle $L$ the angle $\omega = |\tau|$. In this case $(\mathbf{rn}) \leqslant 0$, and therefore it is necessary to prove $\int_L \mathbf{rV}\,dy \leqslant 0$.

**Remark 1.** It is easy to see that result (11) is somewhat narrower than the result obtained in ([2]), where $\theta - \pi \leqslant \tau \leqslant \pi - \omega_0$.

For a multiply connected surface we have the following theorem:

**Theorem 2.** *Let $S$ be a truncated ovaloid with contours $L_1, L_2, \dots, L_n$, which are circles. Suppose that the axes of rotation of these circles meet at one point (which will serve as the origin of coordinates). Then $S$ will be rigid if*

$$\max[0, (\max_L \theta_i - \pi)] \leqslant \tau_i \leqslant \pi - \max[\omega_i, \operatorname{arc\,tg}(\max_L \operatorname{tg}\theta_i + 2\operatorname{tg}\omega_i)] \tag{14}$$

$$(i = 1, 2, \dots, n).$$

**Corollary.** *For sleeve connections of the form (7), a truncated sphere with contours $L_1, \dots, L_n$ will be rigid if*

$$\max[0, \theta_i - \pi] \leqslant \tau_i \leqslant \pi - \max\left[\left(\frac{3\pi}{2} - \theta_i\right), \operatorname{arc\,tg}(\operatorname{tg}\theta_i + 2\operatorname{ctg}\theta_i)\right] \tag{15}$$

$$(i = 1, 2, \dots, n).$$

3. In this section we shall consider separately a sleeve connection of the form $\mathbf{U} \cdot \mathbf{m} = 0$ (on $L$).

Consider a truncated ovaloid with plane contours $L_1, \dots, L_n$; on each contour a sleeve connection of the form $\mathbf{U} \cdot \mathbf{m} = 0$ is imposed, i.e.,

$u_{m_i} = 0$ on $L_i$. Then from (5) it follows that $u_{s_i} = \text{const}$ on $L_i$, and

$$\frac{d\mathbf{V}}{ds} = \left(\frac{dv_{s_i}}{ds} - k_i v_{m_i}\right)\mathbf{s}_i - \left(\frac{u_{s_i}}{\cos\theta_i}\frac{dk_i}{ds}\right)\mathbf{l}_i \equiv H_{l_i}\mathbf{s}_i + G_{l_i}\mathbf{l}_i \quad \text{on } L_i \tag{16}$$

$$(\mathbf{l}_i = \mathbf{n}_i \times \mathbf{s}_i);$$

$$\delta k_i = G_{l_i}\cos\theta_i = -u_{s_i}\frac{dk_i}{ds}. \tag{17}$$

Let us consider two cases: a) one point is fixed on $L_i$, and consequently $u_{s_i} = 0$ on $L_i$; b) $L_i$ is a circle, i.e. $\dfrac{dk_i}{ds} = 0$ on $L_i$. If one of these two cases is realized, then we have

$$\frac{d\mathbf{V}}{ds} \parallel \mathbf{s}_i \quad \text{or} \quad \delta k_i = 0 \quad \text{on } L_i. \tag{18}$$

Using the method of I. N. Vekua ([1]) or N. V. Efimov ([5]), one can prove the following theorem:

**Theorem 3.** *Let $S$ be a truncated ovaloid with plane contours $L_1, \dots, L_n$, where $L_k$, $k = m+1, \dots, n$, are circles. If a bushing perpendicular to the principal normal of this contour is inserted in each aperture, then, in order that $S$ be rigid,* *it is sufficient to fix one point on each contour $L_l$, $l = 1, \dots, m$; moreover, except in the cases when $m = 0$, $n \leq 2$, and the planes of the contours are parallel, $S$ will be kinematically rigid.*

In the case when all contours are circles, this theorem can be proved very easily in the following way.

A simple calculation gives us

$$\int_L \mathbf{rV}\, dV = -\sum_i \left\{ \int_{L_i} \left[ \frac{1}{k_i} \left( \frac{dv_{m_i}}{ds} \right)^2 - k_i v_{m_i}^2 \right] (\mathbf{rb}_i)\, ds + \int_{L_i} v_{b_i} \frac{du_{b_i}}{ds}\, ds \right\}. \tag{19}$$

Choosing the origin of coordinates so that $(\mathbf{rb}_i) \leq 0$ $(= \text{const})$ on the contour $L_i$ $(i = 1, 2, \dots, n)$, by virtue of $v_{b_i} = k_i u_{s_i} = \text{const}$ $(i = 1, \dots, n)$ and Lemma 2 we immediately obtain that $\int_L \mathbf{rV}\, dV \geq 0$. This proves our theorem.

**Remark 2.** Remes and Grotemeyer, in their additions to the work of N. V. Efimov ([6]), also proved the theorem for the case when all contours are circles.

**4.** All the theorems set forth above also hold for the case when $S$ is a piecewise smooth convex surface; it is only required that none of the contours of the surface intersect the lines of fracture of the surface. This will be proved on the basis of the method used by B. Boyarskii and I. N. Vekua in proving the rigidity of a convex closed surface ([7]).

In conclusion I express my heartfelt gratitude to I. N. Vekua for posing the problem and for valuable assistance.

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*Note: Figure translations are in progress. See original paper for figures.*