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Abstract

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MATHEMATICS

L. N. SLOBODETSKII

SPACES OF S. L. SOBOLEV OF FRACTIONAL ORDER AND THEIR APPLICATION TO BOUNDARY-VALUE PROBLEMS FOR PARTIAL DIFFERENTIAL EQUATIONS

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1. Let E_n be the n-dimensional space of points $x = (x_1, \dots, x_n)$; let $n_1, \dots, n_r$ be natural numbers whose sum is n; let $E^{(k)}$ be n_k-dimensional spaces of points

$$x^{(k)} = (x_1^{(k)}, \dots, x_{n_k}^{(k)}) \quad (k = 1, 2, \dots, r).$$

Further, let $\Omega^{(k)}$ be finite or infinite domains in $E^{(k)}$, and

$$Q = \prod_{k=1}^{r} \Omega^{(k)}.$$

We define the function space $W^{(l_k)}_{x^{(k)},2}(Q)$ for nonnegative l_k. First let l_k be an integer. We shall say that $f(x) \in W^{(l_k)}_{x^{(k)},2}(Q)$ if it has generalized derivatives, square-summable over Q, with respect to $x_1^{(k)}, \dots, x_{n_k}^{(k)}$ up to order l_k. The norm of $f(x)$ in $W^{(l_k)}_{x^{(k)},2}(Q)$ is defined to be

$$\|f\|_{W^{(l_k)}_{x^{(k)},2}(Q)} = \left\{ \sum_{q \le l_k} \int_Q |D^q_{x^{(k)}} f|^2 \, dx \right\}^{1/2}.$$

Now let

$$l_k = l_k' + \lambda_k,$$

where l_k' is a nonnegative integer and λ_k is a proper fraction ($0 < \lambda_k < 1$). We shall say that $f(x) \in W^{(l_k)}_{x^{(k)},2}(Q)$ if

$$f(x) \in W^{(l_k')}_{x^{(k)},2}(Q)$$

and if all the integrals

$$L_h^2(D^q_{x^{(k)}} f) = \int_{Q^{(k)}} \left|\Delta(x^{(k)}, y^{(k)}) D^q_{x^{(k)}} f\right|^2 \frac{dx\, dy^{(k)}}{|x^{(k)} - y^{(k)}|^{n_k+2\lambda_k}} \quad (q \leq l'_k),$$

converge, where

$$Q^{(k)} = Q \times \Omega^{(k)} \quad (x \in Q,\ y^{(k)} \in \Omega^{(k)}),$$

$$\Delta(x^{(k)}, y^{(k)}) f = f(x^{(1)}, \dots, x^{(k-1)}, x^{(k)}, x^{(k+1)}, \dots, x^{(r)}) - f(x^{(1)}, \dots, x^{(k-1)}, y^{(k)}, x^{(k+1)}, \dots, x^{(r)}),$$

$$|x^{(k)} - y^{(k)}| = \left[\sum_{s=1}^{n_k} (x_s^{(k)} - y_s^{(k)})^2\right]^{1/2}.$$

In this case we set

$$\|f\|_{W^{(l_k)}_{x^{(k)},2}(Q)} = \left\{\|f\|^2_{W^{(l'_k)}_{x^{(k)},2}(Q)} + \sum_{q \leq l'_k} L_h^2(D^q_{x^{(k)}} f)\right\}^{1/2}.$$

Suppose now that $l_1, \dots, l_r$ are nonnegative numbers. We say that

$$f(x) \in W^{(l_1,\dots,l_r)}_{x^{(1)},\dots,x^{(r)},2}(Q)$$

if

$$f(x) \in W^{(l_k)}_{x^{(k)},2}(Q)$$

for all

$$k = 1, 2, \dots, r.$$

In this case

$$\|f\|_{W^{(l_1,\dots,l_r)}_{x^{(1)},\dots,x^{(r)},2}(Q)} = \left\{\sum_{k=1}^{r} \|f\|^2_{W^{(l_k)}_{x^{(k)},2}(Q)}\right\}^{1/2}.$$

Definition of the spaces $W^{(l_1,\dots,l_r)}_{x(1),\dots,x(r),2}$ is easily generalized to surfaces of the form $\Gamma = S^{(1)} \times \dots \times S^{(r)}$, where $S^{(k)}$ is a sufficiently smooth surface without boundary of dimension m_k $(1 \leq m_k \leq n_k - 1)$, lying in $E^{(k)}$. For $l_1 = l_2 = \dots = l_r = l$, the space $W^{(l_1,\dots,l_r)}_{x(1),\dots,x(r),2}(Q)$ will be denoted by $W_2^{(l)}(Q)$. The functional spaces introduced above are complete Hilbert spaces with an appropriately introduced scalar product.

If the $Q^{(k)}$ are bounded by sufficiently smooth surfaces, then $f(x) \in W^{(l_1,\dots,l_r)}_{x(1),\dots,x(r),2}(Q)$ can be extended to E_n so that its extension $f^*(x) \in W^{(l_1,\dots,l_r)}_{x(1),\dots,x(r),2}(E_n)$ and coincides with $f(x)$ in Q. Therefore, in what follows we shall consider functions defined in all of E_n. To every assertion for $f(x) \in W^{(l_1,\dots,l_r)}_{x(1),\dots,x(r),2}(E_n)$ there will correspond an assertion for $f(x) \in W^{(l_1,\dots,l_r)}_{x(1),\dots,x(r),2}(Q)$.

2. Theorem 1. If $f(x) \in W^{(l_1,\dots,l_r)}_{x(1),\dots,x(r),2}(E_n)$ and $m_1, \dots, m_r$ are nonnegative integers satisfying the inequality

$$\mu_{m_1,\dots,m_r} = 1 - \sum_{k=1}^{r} \frac{m_k}{l_k} > 0,$$

then $f(x)$ has generalized mixed derivatives of the form $D^{m_1}_{x(1)} \cdots D^{m_r}_{x(r)} f \in W^{(\bar{l}_1,\dots,\bar{l}_r)}_{x(1),\dots,x(r),2}(E_n)$, with $\bar{l}_k = \mu_{m_1,\dots,m_r} l_k$ $(k = 1, 2, \dots, r)$. Moreover,

$$\left\| D^{m_1}_{x(1)} \cdots D^{m_r}_{x(r)} f \right\|_{W^{(\bar{l}_1,\dots,\bar{l}_r)}_{x(1),\dots,x(r),2}(E_n)} \le C \|f\|_{W^{(l_1,\dots,l_r)}_{x(1),\dots,x(r),2}(E_n)}, \tag{1}$$

where C does not depend on f.

It follows from this theorem that, for integral l, our space $W^{(l)}_2(Q)$ is equivalent to the corresponding space of S. L. Sobolev.

3. In what follows it is convenient to assume that the $E^{(k)}$ are one-dimensional spaces.

Theorem 2. Let $f(x) \in W^{(l_1,\dots,l_n)}_{x(1),\dots,x(n),2}(E_n)$; $1 \le m \le n-1$; and let $s_{m+1}, \dots, s_n$ be nonnegative integers satisfying the inequality

$$\mu_{s_{m+1},\dots,s_n} = 1 - \sum_{k=m+1}^{n} \frac{s_k}{l_k} - \frac{1}{2} \sum_{k=m+1}^{n} \frac{1}{l_k} > 0. \tag{2}$$

Then on any m-dimensional section E_m of the space E_n by the planes $x_k = c_k$ $(k = m+1, \dots, n)$, the generalized derivatives $D^{s_{m+1}}_{x(m+1)} \cdots D^{s_n}_{x(n)} f \in W^{(\bar{l}_1,\dots,\bar{l}_m)}_{x(1),\dots,x(m),2}(E_m)$, with $\bar{l}_k = \mu_{s_{m+1},\dots,s_n} l_k$ $(k = 1, 2, \dots, m)$. Moreover,

$$\left\| D^{s_{m+1}}_{x(m+1)} \cdots D^{s_n}_{x(n)} f \right\|_{W^{(\bar{l}_1,\dots,\bar{l}_m)}_{x(1),\dots,x(m),2}(E_m)} \le C \|f\|_{W^{(l_1,\dots,l_n)}_{x(1),\dots,x(n),2}(E_n)}, \tag{3}$$

where C does not depend on f and c_k $(k = m+1, \dots, n)$.

Conversely, if for every $s_{m+1}, \dots, s_n$ satisfying inequality (2) functions $\varphi^{(s_{m+1},\dots,s_n)}(x') \in W^{(\bar{l}_1,\dots,\bar{l}_m)}_{x(1),\dots,x(m),2}(E_m)$ are prescribed

$(x' = (x^{(1)}, \dots, x^{(m)}))$, then there exists a function $f(x) \in W^{(l_1,\dots,l_n)}_{x(1),\dots,x(n),2}(E_n)$ satisfying the boundary conditions

$$D^{s_{m+1}}_{x(m+1)} \cdots D^{s_n}_{x(n)} \bar{f} \Big|_{\substack{x_k = c_k \\ k = m+1,\dots,n}} = \varphi^{(s_{m+1},\dots,s_n)}(x') \tag{4}$$

in the sense of strong convergence in $W^{(\bar{l}_1,\dots,\bar{l}_m)}_{x(1),\dots,x(m),2}(E_m)$. Moreover,

$$\|\bar{f}\|_{W^{(l_1,\dots,l_n)}_{x(1),\dots,x(n),2}(E_n)} \le C_1 \sum_s \left\|\varphi^{(s_{m+1},\dots,s_n)}(x')\right\|_{W^{(\bar{l}_1,\dots,\bar{l}_m)}_{x(1),\dots,x(m),2}(E_m)} . \tag{5}$$

Here C_1 does not depend on $\varphi^{(s_{m+1},\dots,s_n)}(x')$.

The theorems obtained have numerous applications in the theory of boundary-value problems for partial differential equations.

4. Consider the polyharmonic equation:

$$\Delta^p u = 0. \tag{6}$$

Let D be a bounded domain in E_n with boundary

$$S = \sum_{m=\beta}^{n-1} S_m,$$

where S_m is a $p+1$ times continuously differentiable surface of dimension m, and β is the greatest m satisfying the inequalities

$$\lambda_m = p - \left[\frac{n-m}{2}\right] - 1 \ge 0, \qquad 1 \le m \le n-1.$$

It is assumed here that the different S_m have no pairwise common points. Further, let $\nu_1, \dots, \nu_{n-m}$ be a complete system of linearly independent normals to S_m. On each S_m define a collection of functions $\varphi^{(l)}_{j_1,\dots,j_l,m}$

$$(m = \beta, \beta+1, \dots, n; \quad l = 0, 1, \dots, \lambda_m; \quad j_1, \dots, j_l = 1, 2, \dots, n-m).$$

It is required to find a function $u = u(x) = u(x_1, \dots, x_n)$ satisfying equation (6) inside D and, on S_m, the boundary conditions

$$\left.\frac{\partial^l u}{\partial \nu_{j_1} \cdots \partial \nu_{j_l}}\right|_{S_m} = \varphi^{(l)}_{j_1,\dots,j_l,m} \tag{7}$$

at least in the sense of weak convergence in L_2 over surfaces parallel to S_m. Using the results of S. L. Sobolev [1] and Theorem 2, we obtain the following proposition.

Theorem 3. In order that problem (6)–(7) be uniquely solvable in $W_2^{(p)}(D)$, it is necessary and sufficient that

$$\varphi^{(l)}_{j_1,\dots,j_l,m} \in W_2^{(\mu_{l,m})}(S_m),$$

where

$$\mu_{l,m} = p - l - \frac{n-m}{2}.$$

When these conditions are fulfilled, the solution satisfies the two-sided inequality

$$C_1 \sum_{m=\beta}^{n-1} \sum_{l=0}^{\lambda_m} \sum_{j_1,\dots,j_l=1}^{n-m} \left\| \varphi^{(l)}_{j_1,\dots,j_l,m} \right\|_{W_2^{(\mu_{l,m})}(S_m)} \le \|u\|_{W_2^{(p)}(D)} \le$$

$$\le C_2 \sum_{m=\beta}^{n-1} \sum_{l=0}^{\lambda_m} \sum_{j_1,\dots,j_l=1}^{n-m} \left\| \varphi^{(l)}_{j_1,\dots,j_l,m} \right\|_{W_2^{(\mu_{l,m})}(S_m)}, \tag{8}$$

where C_1 and C_2 are positive constants depending only on D. Moreover, the boundary conditions (7) are fulfilled in the sense of strong convergence in $W_2^{(\mu_{l,m})}$ over parallel surfaces.

Analogous results can be obtained for boundary-value problems with nonhomogeneous boundary conditions for broad classes of homogeneous

elliptic equations and systems, in particular for strongly elliptic ones.

5. In a domain $D \subset E_n$, bounded by a three-times continuously differentiable surface S, the boundary-value problem is posed: find a solution of the equation

$$\Delta u = f(x), \tag{9}$$

satisfying the boundary condition

$$u\big|_S = \varphi(x). \tag{10}$$

Using Theorem 2 and the results of O. A. Ladyzhenskaya (2), we prove:

Theorem 4. In order that problem (9)–(10) be uniquely solvable in $W_2^{(2)}(D)$, it is necessary and sufficient that $f(x) \in L_2(D)$ and $\varphi \in W_2^{(3/2)}(S)$. Under these conditions, for the solution the two-sided estimate holds

$$C_1 \left[\|f\|_{L_2(D)} + \|\varphi\|_{W_2^{(3/2)}(S)} \right] \le \|u\|_{W_2^{(2)}(D)} \le C_2 \left[\|f\|_{L_2(D)} + \|\varphi\|_{W_2^{(3/2)}(S)} \right], \tag{11}$$

where C_1 and C_2 depend only on D. Moreover, the boundary condition (10) is satisfied in the sense of strong convergence in $W^{(3/2)}(S)$.

Analogous results can be obtained for the boundary-value problem with Neumann-type boundary conditions and for the problem with an oblique derivative. Everything said in this paragraph, with the appropriate changes, extends to boundary-value problems for nonhomogeneous strongly elliptic systems.

6. Let us now consider in $Q = \Omega \times [0, T]$ $(x \in \Omega,\ 0 \le t \le T)$ the heat-conduction equation

$$\frac{\partial u}{\partial t} = \Delta u + f(t, x). \tag{12}$$

For (12) we pose the mixed problem with initial and boundary conditions

$$u\big|_{t=0} = \varphi(x) \quad (x \in \Omega), \qquad u\big|_{\Gamma} = \psi(t, x) \quad (\Gamma = S \times [0, T]). \tag{13}$$

Here S is the three-times continuously differentiable boundary of Ω.

Using Theorem 2 and the results of O. A. Ladyzhenskaya $^{(3)}$, we obtain:

Theorem 5. In order that problem (12)–(13) be uniquely solvable in $W^{(1,2)}_{t,x,2}(Q)$, it is necessary and sufficient that $f(t,x) \in L_2(Q)$, $\varphi(x) \in W^{(1)}_2(\Omega)$, $\psi(t,x) \in W^{(3/4,\,3/2)}_{t,x,2}(\Gamma)$, and that the compatibility condition $\varphi(x)|_S = \psi(0,x)$ be satisfied in the sense of strong convergence in $W^{(1/2)}_2$. Moreover, for the solution the inequality holds

$$C_1 \left[\|\varphi\|_{W^{(1)}_2(\Omega)} + \|\psi\|_{W^{(3/4,\,3/2)}_{t,x,2}(\Gamma)} + \|f\|_{L_2(Q)}\right] \le$$

$$\le \|u\|_{W^{(1,2)}_{t,x,2}(Q)} \le C_2 \left[\|\varphi\|_{W^{(1)}_2(\Omega)} + \|\psi\|_{W^{(3/4,\,3/2)}_{t,x,2}(\Gamma)} + \|f\|_{L_2(Q)}\right], \tag{14}$$

where C_1 and C_2 depend only on Q.

Analogous results are obtained for strongly parabolic systems.

Leningrad State
Pedagogical Institute

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CITED LITERATURE

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