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**Abstract**

**Full Text**

**MATHEMATICS**

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## **THE NEMYTSKII OPERATOR AND ITS POTENTIAL IN ORLICZ SPACES**

*(Presented by Academician S. L. Sobolev, February 7, 1958)*

1. It is known <sup>(1)</sup> that, in the study of nonlinear integral equations of Hammerstein type, an important role is played by the properties of the Nemytskii operator  $h$  and its potential  $f$ . The study of the operator  $h$  and its potential  $f$  in the spaces  $L^p$ , begun by one of the authors <sup>(2)</sup>, led to definitive results <sup>(3)</sup> on the action, boundedness, and continuity of the operator  $h$ , and made it possible to establish the properties of the functional  $f$  in these spaces that are needed for applications <sup>(4)</sup>. In the present work we study the operator  $h$  and its potential  $f$  in Orlicz spaces generalized in the sense of Zaanen <sup>(5)</sup>, <sup>(6-9)</sup>. In Theorem 1, necessary and sufficient conditions are established for the action of the operator  $h$  from some class or Orlicz space into other similar spaces or classes. In the other theorems, conditions are established for the boundedness and continuity of the operator  $h$ , as well as conditions for the continuity and weak lower semicontinuity of the functional  $f$ . The proof of the propositions on the action of the operator  $h$  uses a functional-theoretic construction generalizing the construction used in proving Theorem 1 of <sup>(10)</sup>.

We also note that, although the work is joint, part of the results belongs to one of the authors, and part to the other.

2. Let  $B$  be a set in a finite-dimensional Euclidean space of finite or infinite Lebesgue measure; let  $M(u)$  and  $M_1(u)$  be arbitrary Young functions (see <sup>(8,5,10)</sup>).

Put

$$d = \sup\{u : u \in [0, \infty), M(u) < \infty\}, \quad d_1 = \sup\{u : u \in [0, \infty), M_1(u) < \infty\}.$$

From the definition of Young functions it follows that  $0 < d, d_1 \leq \infty$ . Let the real-valued function  $g(u, x)$  be defined on the topological product  $[-\infty, +\infty] \times B$ , and let  $g(u, x)$  be allowed to take infinite values as well. We assume that  $g(u, x)$  is measurable on  $B$  with respect to  $x$  for every fixed  $u \in (-\infty, +\infty)$ , and is continuous on  $(-\infty, +\infty)$  with respect to  $u$  for almost every fixed  $x \in B$ .

For arbitrary positive  $\lambda, \mu, \nu$  and  $x \in B$ , put:

$$E_x(\lambda, \mu, \nu) = \{u : \lambda u \in \langle -d, d \rangle, M_1[\mu|g(u, x)] > \nu M(\lambda|u|\}),$$

$$F(x; \lambda, \mu, \nu) = \sup\{\mu|g(u, x)| : u \in E_x(\lambda, \mu, \nu) \cup \{0\}\},$$

where  $\langle -d, d \rangle = [-d, d]$  when  $M(d) < \infty$ , and  $\langle -d, d \rangle = (-d, d)$  when  $M(d) = \infty$ .

We note the following properties of the function  $F(x; \lambda, \mu, \nu)$ , which are used in the proof of Theorem 1:

- 1°. The function  $F(x; \lambda, \mu, \nu)$  is measurable in  $x$  for fixed  $\lambda, \mu, \nu$ .
- 2°. For fixed  $x, \lambda, \mu$ , it is nonincreasing in  $\nu$ .
- 3°. For fixed  $x, \mu, \nu$ , it is nonincreasing in  $\lambda$ .
- 4°. For fixed  $x, \lambda, \nu$ , it is nondecreasing in  $\mu$ .

3. Consider the Nemytskii operator  $h: hu = g(u(x), x)$ . Let  $L^M = L^M(B)$  and  $L^{M_1} = L^{M_1}(B)$  be the Orlicz spaces generated by the Young functions  $M(u)$  and, respectively,  $M_1(u)$ . As is known (see (8), p. 100, and (5)), the space  $L^M$  consists of all functions  $u(x)$ , measurable on  $B$ , for which

$$\int_B M[k|u(x)|] dx < \infty, \quad (1)$$

where  $k$  is a certain positive number, depending on the function  $u(x)$ . The functions  $u(x)$  satisfying inequality (1) with  $k = 1$  form the class  $L_M$ . The functions  $u(x)$  satisfying inequality (1) for every  $k$  form the subspace  $L_M^X$  of the space  $L^M$  (11) (see also (9)). The space  $L_M^X$  is of interest only in the case  $d = \infty$ , since for  $d < \infty$  it contains only the zero element. In view of this, everywhere that the action of the operator  $h$  from  $L_M^X$  into some class or space is discussed, we shall assume that  $d = \infty$ .

We shall say that the operator  $h$  acts from  $(L^M, L_M, L_M^X)$  into  $(L^{M_1}, L_{M_1}, L_{M_1}^X)$  if it acts from some class indicated in the first parentheses into some class indicated in the second parentheses.

The following proposition establishes a characteristic property of the function  $g(u, x)$  that generates the operator  $h$  from  $(L^M, L_M, L_M^X)$  into  $(L^{M_1}, L_{M_1}, L_{M_1}^X)$ .

**Theorem 1.** In order that  $h$  act from  $(L^M, L_M, L_M^X)$  into  $(L^{M_1}, L_{M_1}, L_{M_1}^X)$ , it is necessary and sufficient that the conditions indicated in the following table, respectively, be satisfied:

| from / into | $L^{M_1}$   | $L_{M_1}$  | $L_{M_1}^X$   |
|-------------|---|--|---|
| $L^M$       | For every $\lambda$ there exist $\mu$ and $\nu$ such that $F(x; \lambda, \mu, \nu) \in L_{M_1}$ | For every $\lambda$ there exists $\nu$ such that $F(x; \lambda, 1, \nu) \in L_{M_1}$ | For any $\lambda$ and $\mu$ there exists $\nu$ such that $F(x; \lambda, \mu, \nu) \in L_{M_1}$  |
| $L_M$       | There exist $\mu$ and $\nu$ such that $F(x; 1, \mu, \nu) \in L_{M_1}$                           | There exists $\nu$ such that $F(x; 1, 1, \nu) \in L_{M_1}$                           | For every $\mu$ there exists $\nu$ such that $F(x; 1, \mu, \nu) \in L_{M_1}$                    |
| $L_M^X$     | There exist $\lambda, \mu,$ and $\nu$ such that $F(x; \lambda, \mu, \nu) \in L_{M_1}$           | There exist $\lambda$ and $\nu$ such that $F(x; \lambda, 1, \nu) \in L_{M_1}$        | For every $\mu$ there exist $\lambda$ and $\nu$ such that $F(x; \lambda, \mu, \nu) \in L_{M_1}$ |

Let  $\mathfrak{F}$  be a family of functions measurable on  $B$  with the following property: if  $u(x) \in \mathfrak{F}$  and a function  $v(x)$ , measurable on  $B$ , satisfies the condition  $|v(x)| \leq |u(x)|$  almost everywhere, then  $v(x) \in \mathfrak{F}$ . Examples of such families are  $L^{M_1}, L_{M_1}, L_{M_1}^X$ .

**Theorem 2.** Let  $d < \infty$ . Then, in order that the operator  $h$  act from  $L^M$  into  $\mathfrak{F}$ , it is sufficient, and if  $L^M$  consists of the same functions as  $L^\infty$ , then also necessary, that the following condition be satisfied:

$$\sup_{|u| \leq \alpha} |g(u, x)| \equiv a_\alpha(x) \in \mathfrak{F}$$

for every nonnegative  $\alpha$ .

Let us also note the following proposition.

**Theorem 3.** Let  $d = \infty$  and  $d_1 < \infty$ . Then, in order that the operator  $h$  act from any one of the classes  $L^M, L_M, L_M^X$  into  $L^{M_1}$ , it is necessary, and if  $L^{M_1}$  consists of the same functions as  $L^\infty$ , then also sufficient, that the following condition be satisfied: there exists a constant  $C$  such that, for almost all  $x \in B$ ,  $|g(u, x)| \leq C$  for  $u \in (-\infty, +\infty)$ .

This theorem generalizes the proposition from (3) on the action of the operator  $h$  from  $L^p$  into  $L^\infty$ .

- In (3) a necessary and sufficient condition for the action of the operator  $h$  from the class  $L^p$  into the class  $L^{p_1}$  was published in the form of a certain inequality for the function  $g(u, x)$ . Similar conditions are obtained from Theorem 1 also for generalized Orlicz spaces. We give two such

propositions. Put

$$M_1^{-1}(v) = \sup\{u : u \geq 0, M_1(u) \leq v\}, \quad v \geq 0.$$

$M_1^{-1}(v)$  is a nondecreasing and continuous function on  $[0, \infty)$ .

**Theorem 4.** In order that the operator  $h$  act from  $L^M$  into  $L^{M_1}$ , it is necessary and sufficient that the following condition be satisfied: to each positive number  $\lambda$  there correspond a function  $a_\lambda(x) \in L^{M_1}$  and positive numbers  $\alpha_\lambda$  and  $\beta_\lambda$  such that

$$|g(u, x)| \leq a_\lambda(x) + \alpha_\lambda M_1^{-1}[\beta_\lambda M(\lambda|u|)]$$

for all  $x \in B$ ,  $\lambda u \in \langle -d, d \rangle$ .

It is said <sup>(6)</sup> that the function  $M_1(u)$  satisfies the  $\Delta_2$ -condition if  $d_1 = \infty$  and  $M_1(2u) \leq CM_1(u)$  for  $u \geq u_0 \geq 0$ , where  $C = \text{const}$ . In what follows, when the  $\Delta_2$ -condition is discussed, we shall assume  $u_0 = 0$  if  $\text{mes } B = \infty$ .

**Theorem 5.** In order that the operator  $h$  act from  $L_M$  into  $L_{M_1}$ , it is necessary, and if  $M_1(u)$  satisfies the  $\Delta_2$ -condition then also sufficient, that for all  $x \in B$ ,  $u \in \langle -d, d \rangle$  the inequality

$$|g(u, x)| \leq a(x) + M_1^{-1}[bM(|u|)], \quad (2)$$

hold, where  $a(x) \in L_{M_1}$ ,  $b > 0$ .

This theorem generalizes the proposition from <sup>(3)</sup> on the action of the operator  $h$  from  $L^p$  into  $L^{p_1}$ , when  $p_1 < \infty$ . Propositions analogous to Theorems 4 and 5 hold also in the other cases of Theorem 1.

**Remark.** If in Theorem 5 inequality (2) is replaced by the inequality

$$|g(u, x)| \leq \max\{a(x), M_1^{-1}[bM(|u|)]\},$$

then the reservation about the  $\Delta_2$ -condition for the function  $M_1(u)$  is not needed.

5. Let us now consider the question of boundedness and continuity of the operator  $h$  (cf. <sup>(12)</sup>).

**Theorem 6.** If the operator  $h$  acts from  $Z$  into  $L^{M_1}$ , where  $Z$  coincides with one of the classes:  $L^M$ ,  $L_M$ ,  $L_M^\lambda$ , then it is bounded on  $S$ , where  $S$  coincides with  $L^M$  for  $Z = L^M$ , with the unit ball  $\|u\|_M \leq 1$  of the space  $L^M$  for  $Z = L_M$ , and with the ball  $D = \{u(x) : \|u\|_M = \lambda^{-1}\}$  for  $Z = L_M^\lambda$  ( $\lambda$  is the number appearing in Theorem 1 in the case of the action of the operator  $h$  from  $L_M^\lambda$  into  $L^{M_1}$ ).

**Theorem 7.** If the operator  $h$  acts from  $L^M$  into  $L_{M_1}^\lambda$ , or from  $L_M^\lambda$  into  $L_{M_1}^\lambda$ , then it is continuous.

We note that the following proposition also holds: if the operator  $h$  acts from some ball of the space  $L^M$  into  $L_{M_1}^\lambda$ , then it is continuous at every interior point of this ball.

6. Let the operator  $h$  act from  $L^M$  into  $L^{M_1}$ , where  $M(u)$  and  $M_1(u)$  are mutually complementary Young functions (see (5) and (8), p. 99). As in (4), consider the functional

$$f(u) = \int_B dy \int_0^{u(y)} g(v, y) dv, \quad (3)$$

for which the following propositions hold.

**Theorem 8.** If the operator  $h$  acts from  $L^M$  into  $L^{M_1}$ , then the functional  $f(u)$  satisfies the Lipschitz condition in every ball of the space  $L^M$ .

**Theorem 9.** If the operator  $h$  acts from  $L^M$  into  $L^{M_1}$ , then it is the gradient of the functional (3):  $\text{grad } f(u) = hu$ , i.e.  $f$  is a potential of the operator  $h$ .

Analogous propositions hold for the functional  $f(u)$  when the operator  $h$  acts from  $L_M$  or  $L_M^X$  into  $L^{M_1}$ .

7. The continuity of the functional (3) is used when, by the variational method (1), one proves the existence of eigenfunctions of the Hammerstein operator  $\Gamma = Ah$ , or of solutions of the equations  $\Gamma u = u$ , where  $A$  is a linear integral operator. In this case one must require that  $A$  be a completely continuous operator from  $L^{M_1}$  or  $L_{M_1}^X$  into  $L^M$ . If, however, the functional  $f(u)$  is weakly semicontinuous, then in proving the existence of eigenfunctions of the operator  $\Gamma$ , or of solutions of the equation  $\Gamma u = u$ , one may dispense with the requirement of complete continuity of the operator  $A$ , replacing it by the requirement that the operator  $A$  be bounded. In view of this, it is important to establish conditions for the weak semicontinuity of the functional (3). We give one such proposition.

**Theorem 10.** Let the function  $g(u, x)$ , generating the operator  $h$  from  $L^M$  into  $L^{M_1}$  or into  $L_{M_1}^X$ , be nonincreasing (nondecreasing) with respect to  $u$  for almost every fixed  $x \in B$ . Then the functional  $f(u)$ , defined by equality (3), is weakly lower (upper) semicontinuous in  $L^M$ .

8. In conclusion, we note that the propositions established in the present work, by using topological or variational methods, lead to new criteria for the existence both of eigenfunctions of the operator  $\Gamma$  and of solutions of the equation  $\Gamma u = u$ .

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## REFERENCES

1. M. M. Vainberg, *Variational Methods for the Study of Nonlinear Operators*, 1956.
2. M. M. Vainberg, DAN, 61, No. 6 (1948); *Uspekhi Mat. Nauk*, 4, issue 3 (31), 130 (1949); *Matem. Sbornik*, 26 (68), 3 (1950).
3. M. M. Vainberg, DAN, 73, No. 2 (1950); DAN, 92, No. 2 (1953).
4. M. M. Vainberg, *Uspekhi Mat. Nauk*, 7, issue 4 (1952).
5. A. C. Zaanen, *Kon. Nederl. Akad. Wetensch. Proc.*, 52, No. 5-6 (1949); *Linear Analysis*, Amsterdam—N. Y. 1953.
6. Z. Birnbaum, W. Orlicz, *Stud. Math.*, 3, 1 (1931).
7. W. Orlicz, *Bull. Intern. Acad. Polon.*, ser. A, Cracovie, No. 8-9 (1932); No. 3-4 (1936).
8. A. Zygmund, *Trigonometric Series*, 1939.
9. M. A. Krasnosel'skii, Ya. B. Rutitskii, *Transactions of the Seminar on Functional Analysis*, Voronezh, issue 1 (1956).
10. I. V. Shragin, DAN, 117, No. 1 (1957).
11. W. A. J. Luxemburg, *Banach Function Spaces*, Thesis Delft Techn. Univ., Assen, 1955.
12. Ya. B. Rutitskii, *Reports of the Academy of Sciences of the Ukrainian SSR*, No. 3 (1952).

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