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Abstract

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HARMONIC QUADRICS AND CERTAIN RULED SURFACES OF CONGRUENCES

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1. In the present note we consider congruences whose harmonic ruled surfaces form quadrics, and in connection with this several theorems are proved. Several new ruled surfaces of congruences are also considered. The method of Cartan exterior forms is used in the paper (^{1}).

2. The infinitesimal displacement of the tetrad $\{A_i\}$ is defined by the equations $dA_i = \omega_i^k A_k$ $(i, k = 1, 2, 3, 4)$, where $D\omega_i^k = [\omega_j^i \omega_i^k]$. A first-order tetrad is defined by the differential equations $\omega_1^4 = 0$, $\omega_2^3 = 0$. For the differential neighborhood up to the 5th order the following formulas are obtained (for details on these formulas see (^{2}), pp. 344—349):

$$\omega_3^4 = \alpha\omega_1^3 - \beta\omega_2^4, \qquad \omega_1^2 = \beta\omega_1^3 + \gamma\omega_2^4,$$

$$\Delta\alpha = \alpha_1\omega_1^3 - \beta_1\omega_2^4, \qquad \Delta\beta = \alpha\beta_1\omega_1^3 + \gamma\beta_2\omega_2^4, \qquad \Delta\gamma = \beta_2\omega_1^3 + \gamma_2\omega_2^4, \quad \text{(A)}$$

$$\Delta\alpha_1 = \alpha_{11}\omega_1^3 - \beta_{11}\omega_2^4, \qquad \Delta\beta_1 = \beta_{11}\omega_1^3 + \gamma\beta_{12}\omega_2^4,$$

$$\Delta\beta_2 = \alpha\beta_{12}\omega_1^3 + \beta_{22}\omega_2^4, \qquad \Delta\gamma_2 = \beta_{22}\omega_1^3 + \gamma_{22}\omega_2^4,$$

and the formulas obtained from these by replacing the indices: 1 by 2; 3 by 4, and by adding primes to the coefficients α, β, γ with arbitrary indices.

3. The conditions which the given congruence and its ruled surfaces $\omega_2^4 = \lambda\omega_1^3$ must satisfy in order that the latter be quadrics are written in the form:

$${}^3/_2(\lambda_1 + \lambda\lambda_2)(\alpha'\lambda^2 + \gamma') - \gamma'\gamma_1' + \alpha'\alpha_2'\lambda^3 - 3\lambda(\gamma'\beta_1 + \lambda\alpha'\beta_2') = 0,$$

$${}^3/_2(\lambda_1 + \lambda\lambda_2)(\gamma\lambda^2 + \alpha) - \alpha\alpha_1 + \gamma\gamma_2\lambda^3 + 3\lambda(\alpha\beta_1 + \lambda\gamma\beta_2) = 0, \qquad (1)$$

$$\lambda(\lambda_{11} + \lambda\lambda_{12}) + \lambda^2(\lambda_{21} + \lambda\lambda_{22}) + {}^1/_2\lambda(\lambda_2^2 - \lambda_1^2) +$$

$$+(\gamma\lambda^2 + \alpha)(\lambda^2\alpha' - 2\lambda\beta' - \gamma') + (\alpha'\lambda^2 + \gamma')(\lambda^2\gamma + 2\lambda\beta - \alpha) = 0,$$

where

$$d\ln\lambda + \omega_1^1 + \omega_4^4 - \omega_2^2 - \omega_3^3 = \lambda_1\omega_1^3 + \lambda_2\omega_2^4,$$

$$d\lambda_1 + \lambda_1(\omega_1^1 - \omega_3^3) + 2\omega_3^1 = \lambda_{11}\omega_1^3 + \lambda_{12}\omega_2^4, \qquad (2)$$

$$d\lambda_2 + \lambda_2(\omega_2^2 - \omega_4^4) - 2\omega_4^2 = \lambda_{21}\omega_1^3 + \lambda_{22}\omega_2^4.$$

The harmonic ruled surfaces of the first and second focal surfaces of the congruence are written respectively by the equations $(^{3-5})$

$$\alpha(\omega_1^3)^2 - \gamma(\omega_2^4)^2 = 0, \qquad \alpha'(\omega_2^4)^2 - \gamma'(\omega_1^3)^2 = 0.$$

For harmonic ruled surfaces of the first focal surface,

$$\lambda = \varepsilon\sqrt{\alpha/\gamma} \quad (\varepsilon = \pm 1). \qquad (3)$$

Differentiating equation (3), by virtue of (A) and (2) we obtain

$$2\lambda_1 = \alpha_1 - \beta_2, \qquad -2\lambda_2 = \beta_1 + \gamma_2. \qquad (4)$$

A new differentiation of system (4), with the aid of (A) and (2), leads to the equations

$$\begin{gathered} 2\lambda_{11} = \alpha_{11} - \alpha\beta_{12}, \qquad 2\lambda_{12} = 3\alpha\alpha' + \gamma\gamma' - \beta_{11} - \beta_{22} - 4\beta\beta', \\ 2\lambda_{12} = 4\beta\beta' - \alpha\alpha' - 3\gamma\gamma' - \beta_{11} - \beta_{22}, \qquad 2\lambda_{22} = -\gamma_{22} - \gamma\beta_{12}. \end{gathered} \qquad (5)$$

If both harmonic ruled surfaces of the first focal surface (A_1) are quadrics, then the obtained values for $\lambda, \lambda_1, \lambda_2, \lambda_{12}, \lambda_{21}, \lambda_{11}, \lambda_{22}$, independently of ε, must satisfy system (1), and then we obtain

$$\begin{gathered} \alpha_1 + 3\beta_2 = 0, \qquad \gamma_2 - 3\beta_1 = 0, \qquad \alpha_2' + 3\beta_1' = 0, \qquad \gamma_1' - 3\beta_2' = 0, \\ \alpha\alpha' - \gamma\gamma' = 0, \qquad \beta_1 + \beta_1' = 0, \qquad \beta_2 + \beta_2' = 0, \qquad \beta_{12} + \beta' = 0. \end{gathered} \qquad (6)$$

The invariant equations (6) characterize our congruence. The Laplace sequence generated by this congruence is called a configuration L. This configuration has the following properties:

1) The Laplace sequence closes after the fourth step and is an R-sequence (^{6}).
2) All focal surfaces of this sequence are quadrics, and focal surfaces situated at opposite vertices of the sequence coincide.
3) All four congruences of the sequence are equivalent.
4) Two diagonals of the sequence describe one linear congruence.
5) The two harmonic ruled surfaces of one congruence of the configuration L coincide with the harmonic ruled surfaces of the remaining congruences of the sequence.

All these results make it possible to formulate a number of theorems.

Theorem 1. *Both harmonic ruled surfaces of one focal surface are quadrics if and only if this congruence generates a configuration* L.

Theorem 2. *Two harmonic quadrics of one congruence of a configuration* L *are simultaneously harmonic quadrics also for the remaining congruences of the configuration.*

If both asymptotic ruled surfaces of one focal surface of a congruence are quadrics, then the Laplace sequence generated by this congruence is an R-sequence; all focal surfaces and asymptotic ruled surfaces of each congruence of the sequence are also quadrics. This sequence need not close after the fourth step; consequently, the configuration L is a special case of this sequence (when $\beta_{12} + \beta' = 0$). This circumstance makes it possible to state Theorem 3.

Theorem 3. *If both harmonic ruled surfaces of a given focal surface of a congruence are quadrics, then all asymptotic ruled surfaces of this congruence are also quadrics.*

The converse theorem is false, but, obviously, the following theorem is true:

Theorem 4. *If both asymptotic ruled surfaces of one focal surface of a congruence are quadrics and the sequence*

···surface R generated by this congruence closes up after the fourth step, then the harmonic ruled surfaces of this congruence are also quadrics.

Thus, through each ray of every congruence of the configuration L there pass four demiquadrics belonging to this configuration: two asymptotic and two harmonic ruled surfaces. Of these two pairs, one pair (either the first or the second) is always real, and the other imaginary.

4. In addition to asymptotic and harmonic ruled surfaces, congruences have infinitely many other remarkable ruled surfaces. Let us clarify the geometric meanings of these ruled surfaces.

In the note (^{3}) the equation of Lie' s quadric of a ruled surface ($\omega_2^4 = \lambda\omega_1^3$) of the congruence was obtained. It has the form

$$2\lambda(x^1x^4 - \lambda x^2x^3) - \lambda(\lambda_1 + \lambda\lambda_2)x^3x^4 + \\ +(\lambda^2\alpha' - 2\lambda\beta' - \gamma')x^4x^4 + \lambda(\lambda^2\gamma + 2\lambda\beta - \alpha)x^3x^3 = 0. \tag{7}$$

The tangent plane of the focal surface (A_1) intersects the surface (7) in two generators $x_4 = 0,\ x_3 = 0$ and

$$x_4 = 0, \qquad -2\lambda x^2 + (\lambda^2\gamma + 2\lambda\beta - \alpha)x^3 = 0. \tag{8}$$

Choose λ so that the generator (8) is tangent to the harmonic lines $\omega_2^4 = \varepsilon\sqrt{\frac{\alpha}{\gamma}}\,\omega_1^3$ of the focal surface (A_1). Since the tangent to the harmonic lines intersects the edge (A_2A_3) at the point $(\beta + \varepsilon\sqrt{\alpha\gamma})A_2 + A_3$, the generator (8) will also pass through this point if and only if

$$\lambda = \sqrt{\frac{\alpha}{\gamma}}\left(\varepsilon + \varepsilon_1\sqrt{2}\right), \qquad \varepsilon_1 = \pm 1, \qquad \varepsilon = \pm 1. \tag{9}$$

Thus, with each harmonic ruled surface

$$\omega_2^4 = \varepsilon\sqrt{\frac{\alpha}{\gamma}}\,\omega_1^3$$

there are associated two new ruled surfaces

$$\omega_2^4 = \sqrt{\frac{\alpha}{\gamma}}\left(\varepsilon + \varepsilon_1\sqrt{2}\right)\omega_1^3.$$

In what follows we shall denote the harmonic ruled surfaces by G, and the new ruled surfaces by G_1. The analogous ruled surfaces for the focal surface (A_2) will be denoted by the same symbols with primes.

Now choose λ so that the generator (8) is tangent to the net G_1. Since the tangent to the net G intersects the edge A_2A_3 at the point

$$[\beta + (\varepsilon + \varepsilon_1\sqrt{2})\sqrt{\alpha\gamma}]A_2 + A_3,$$

this point will lie on the line (8) if and only if

$$\lambda = \sqrt{\frac{\alpha}{\gamma}}\left(\varepsilon + \varepsilon_1\sqrt{2} + 4\varepsilon_2 + 2\varepsilon_1\varepsilon_2\sqrt{2}\right), \qquad \varepsilon = \pm 1, \quad \varepsilon_1 = \pm 1, \quad \varepsilon_2 = \pm 1. \tag{10}$$

The ruled surfaces corresponding to these lines will be denoted by G_2. Continuing this process, we shall obtain ever newer ruled surfaces of the congruence $G_3, G_4, \ldots$. Each G_i contains 2^{i+1} new ruled surfaces. Obviously, in the congruence W all ruled surfaces G_i coincide with the ruled surfaces G'_i for any number i.

It is known that the net G on the focal surface (A_1) is a conjugate net. We shall prove here that the lines G_1 are formed from two conjugate nets.

For the proof, choose the third vertex of the coordinate tetrahedron on the second tangent focal net (A_i). This choice is characterized by the equation (see (2), p. 360) $\beta = 0$. The tangents to the lines G_1 intersect the edge A_2A_3 at four points

$$M_\varepsilon = (\varepsilon + \sqrt{2})\sqrt{\alpha\gamma}A_2 + A_3, \qquad N_\varepsilon = (\varepsilon - \sqrt{2})\sqrt{\alpha\gamma}A_2 + A_3 \qquad (\varepsilon = \pm 1).$$

The tangents to the asymptotic lines of the focal surface intersect the edge A_2A_3 at two points:

$$A = \sqrt{-\alpha\gamma}\,A_2 + A_3, \qquad B = -\sqrt{-\alpha\gamma}\,A_2 + A_3. \tag{10'}$$

The cross ratio of the four points $M_\varepsilon, N_\varepsilon, A, B$ is written in the form

$$(M_\varepsilon, N_\varepsilon, A, B) = \frac{\sqrt{-1} - \varepsilon - \sqrt{2}}{\varepsilon - \sqrt{2} - \sqrt{-1}} \cdot \frac{\varepsilon - \sqrt{2} + \sqrt{-1}}{-\sqrt{-1} - \varepsilon - \sqrt{2}} = -1,$$

i.e., these points form a harmonic quadruple independently of ε, as was to be proved.

Now we prove that *each family of lines G_i on the focal surface A_1 is formed from 2^i conjugate nets.*

This theorem will be obvious if we prove the following theorem: *There exist two ruled surfaces whose Lie quadrics intersect the given focal plane along a preassigned straight line. Two lines on this focal surface corresponding to these ruled surfaces are conjugate to one another.*

Indeed, suppose that on the first tangent plane we have an arbitrary straight line determined by two points A_1 and $(\beta + \alpha\gamma)A_2 + A_3$, where a is an arbitrarily prescribed quantity. The generator (8) coincides with this line if and only if

$$\lambda^2\gamma - 2a\gamma\lambda - \alpha = 0, \tag{11}$$

i.e., we have obtained a quadratic equation in λ; consequently, the first part of the theorem is proved. The tangents to the lines $\omega_2^4 = \lambda_1\omega_1^3$ and $\omega_2^4 = \lambda_2\omega_1^3$ (λ_1, λ_2 are the roots of equation (11)) intersect the edge A_2A_3 at the points (for $\beta = 0$)

$$M = \gamma\left(a + \sqrt{a^2 + \frac{\alpha}{\gamma}}\right)A_2 + A_3, \qquad N = \gamma\left(a - \sqrt{a^2 + \frac{\alpha}{\gamma}}\right)A_2 + A_3.$$

These two points, together with the two points A and B (10′) of the asymptotic tangents, form a harmonic quadruple; i.e., the two lines $\omega_2^4 = \lambda_1\omega_1^3$ and $\omega_2^4 = \lambda_2\omega_1^3$ on the surface A_1 form a conjugate net.

The question of when the ruled surfaces G_i are quadrics remains open.

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Note: Figure translations are in progress. See original paper for figures.

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