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Abstract

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A CRITERION FOR SEMI-REDUCIBILITY OF HOMOGENEOUS RIEMANNIAN SPACES

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Consider a proper Riemannian ($ds^2 > 0$) homogeneous space V_n with a continuous group of motions G_r and the stationary subgroup H of a given point M. Suppose that H, regarded as a group of rotations in the Euclidean tangent space E_n at the point M, decomposes into the direct product of two subgroups:

$$H = H_0 \times H_1,$$

acting, respectively, on the planes E_q and E_{n-q}, which are, obviously, mutually orthogonal*.

The purpose of the present note is to establish to what extent, from the indicated decomposition of the stationary group H, there follows a foliation of the whole space V_n into mutually complementary and orthogonal surfaces V_q and V_{n-q}.

The following theorem, proved by Wakakuwa, is known ([1]):

Theorem 1. *If H_0 fixes no one-dimensional direction in the plane E_q, and H_1 fixes none in E_{n-q}, then the space V_n is reducible:*

$$V_n = V_q \times V_{n-q},$$

i.e. it is foliated into mutually orthogonal totally geodesic surfaces V_q and V_{n-q}.

We shall show that, under certain restrictions on the group H, the space V_n is semi-reducible, i.e. it decomposes into mutually orthogonal surfaces V_q and V_{n-q}, of which some are totally geodesic, while the others are umbilical and similar to one another ([2]). Here it is essential that the restrictions are imposed on only one of the components of the group H; the other remains completely arbitrary.

Choose at the point M an orthonormal frame

$$R_0(M, \bar{e}_i, \bar{e}_\alpha)^{**}$$

so that the $\bar{e}_i$ form a basis in E_q, and the $\bar{e}_\alpha$ in E_{n-q}. Subjecting the frame R_0 to all possible transformations of the group G_r, we obtain a family of frames R, depending on the n coordinates of the point M and on the $r - n$ parameters of the group H, and invariant with respect to G_r. The connection of the space V_n, referred to these frames, as is known, is determined by the formulas

$$dM = \omega^a \bar{e}_a, \qquad d\bar{e}_a = \omega_a^b \bar{e}_b \quad (\omega_a^b + \omega_b^a = 0),$$

where the forms ω^a contain only the differentials dx^a. Moreover, in view of the invariance of the planes E_q and E_{n-q} with respect to rotations from H, one must have

$$\omega_i^\alpha = C_{ia}^\alpha \omega^a \quad (C_{i\alpha}^\alpha = -C_{\alpha\alpha}^i) \tag{1}$$

* The case is not excluded in which one of the subgroups, for example H_0, is trivial, i.e. coincides with the identity transformation on the plane E_q.

** The indices everywhere range over the following limits: $a, b, c = 1, \ldots, n;\; i, j, k = 1, \ldots, q;\; \alpha, \beta, \gamma = q+1, \ldots, n.$

with constant C_{ia}^α (since $\omega_i^\alpha, \omega^a$ are invariants with respect to the transitive group G_r).

Under rotations of the group H, the forms ω^a, ω_b^a receive increments:

$$\delta\omega^a = L_b^a \omega^b \delta t, \qquad \delta\omega_b^a = (L_c^a \omega_b^c - L_b^c \omega_c^a)\delta t,$$

where $L = \|L_b^a\|$ is one of the matrices of infinitesimal rotations. Applying these formulas to (1) separately for rotations L from H_0 and L from H_1, we obtain:

$$\begin{array}{ll} \text{a) } \underset{1}{L}{}_\beta^\alpha C_{ik}^\beta = 0, & \text{b) } \underset{1}{L}{}_\beta^\alpha C_{i\gamma}^\beta = C_{i\beta}^\alpha \underset{1}{L}{}_\gamma^\beta; \\ \text{c) } -\underset{0}{L}{}_i^j C_{jk}^\alpha = C_{ij}^\alpha \underset{0}{L}{}_k^j, & \text{d) } \underset{0}{L}{}_i^j C_{j\gamma}^\alpha = 0. \end{array} \tag{2}$$

Let us further assume that one of the components of the group H, for example H_1, is irreducible. In particular, H_1 fixes no one-dimensional directions in E_{n-q}, and therefore from equality a) it follows that $C_{ik}^\beta = -C_{\beta k}^i = 0$, while c) is identically satisfied. If we denote $C_k = \|C_{k\beta}^\alpha\|$, then b) in (2) can be written briefly as

$$\underset{1}{L} C_k = C_k \underset{1}{L}. \tag{3}$$

Lemma. *If a matrix C commutes with an irreducible group of orthogonal matrices, then its symmetric part is a scalar matrix (i.e., of the form λE), and, in the case when the dimension of the space is odd, its skew-symmetric part is the zero matrix.*

On the basis of this lemma the elements of the matrix C_k have the form:

$$C^{\alpha}_{k\beta} = A^{\alpha}_{k\beta} + \lambda_k \delta^{\alpha}_{\beta}, \qquad A^{\alpha}_{k\beta} = -A^{\beta}_{k\alpha}, \qquad \delta^{\alpha}_{\beta} = \begin{cases} 0, & \alpha \neq \beta, \\ 1, & \alpha = \beta. \end{cases}$$

Using these formulas in the structure equations of the space V_n under consideration, we obtain:

$$D\omega^i = [\omega^k \omega^i_k] - A^{\alpha}_{i\beta}[\omega^{\alpha}\omega^{\beta}],$$

$$D\omega^{\alpha} = [\omega^{\beta}\omega^{\alpha}_{\beta}] + A^{\alpha}_{k\beta}[\omega^k \omega^{\beta}] + [\lambda_k \omega^k \omega^{\alpha}]. \tag{4}$$

It is clear from this that the system of equations $\omega^{\alpha} = 0$ is completely integrable, since the exterior differentials of the forms ω^{α} vanish as a consequence of the system itself. As for the system $\omega^i = 0$, it is completely integrable if and only if all $A^{\alpha}_{k\beta}$ are equal to zero. This is satisfied, according to the lemma, if the plane E_{n-q} is odd-dimensional. If, however, $n - q$ is even, then we additionally assume that $A^{\alpha}_{k\beta} = 0$. This is equivalent to the requirement that the group H_1 admit no rotation commuting with it.

From the complete integrability of the systems $\omega^{\alpha} = 0$ and $\omega^i = 0$, it follows that the planes E_q and E_{n-q} are holonomic, i.e., they envelope the surfaces V_q and V_{n-q}. Let us pass to a new coordinate system determined by the orthogonal decomposition of V_n into these surfaces, and denote by the symbol d differentiation with respect to x^i, and by δ differentiation with respect to x^{α}. Then, first,

$$\omega^i = \omega^i(d), \qquad \omega^{\alpha} = \omega^{\alpha}(\delta),$$

and, secondly, from the structure equations (4), taking into account $A^{\alpha}_{k\beta} = 0$:

$$\delta \sum_i (\omega^i(d))^2 = 0,$$

$$d \sum_{\alpha} (\omega^{\alpha}(\delta))^2 = -2\lambda_k \omega^k(d) \sum_{\alpha} (\omega^{\alpha}(\delta))^2, \tag{5}$$

$$\delta(\lambda_k \omega^k(d)) = 0.$$

To prove the last equality, the equations d) from (2) were also used, from which it follows that $\overset{0}{L}{}^i_j \lambda_i = 0$, i.e., the λ_i determine one-dimensional invariant directions with respect to H_0 on the plane E_q.

It follows immediately from (5) that the metric ds^2 in the indicated coordinate system has the form

$$ds^2 = \sum_i (\omega^i)^2 + \sum_\alpha (\omega^\alpha)^2 = g_{ij}(x^k)\, dx^i dx^j + \sigma(x^k)\, a_{\alpha\beta}(x^\gamma)\, dx^\alpha dx^\beta, \qquad (6)$$

and, consequently, the space V_n is semireducible ([2]). The group G_r is an unmixed group ([3]) with respect to the decomposition (6); otherwise we would arrive at a contradiction with the assumption of irreducibility of H_1.

Thus, the following theorem has been proved:

Theorem 2. Let there be given a homogeneous Riemannian space V_n ($ds^2 > 0$) with group G_r and stationary subgroup H. Suppose that the group H decomposes into the direct product of two subgroups, one of which is irreducible, and if the plane on which it acts in the tangent space of the given point is four-dimensional, it has no mutually permutable rotations. Then the space V_n is semireducible, and its metric is reduced to the form (6), with respect to which the group G_r is unmixed.

From Theorem 2 there follows a more general theorem:

Theorem 3. If in a homogeneous proper Riemannian space V_n the stationary group

$$H = H_0 \times H_1 \times \dots \times H_p,$$

and all groups H_t ($t > 0$) possess the property indicated in Theorem 2, then the space V_n is ρ-fold semireducible, i.e., in some coordinate system

$$ds^2 = ds_0^2(x^i) + \sigma_1(x^i)\, ds_1^2(x^{\alpha_1}) + \dots + \sigma_p(x^i)\, ds_p^2(x^{\alpha_p}). \qquad (7)$$

With respect to the decomposition (7), the group G_r is unmixed.

Remark 1. The function σ in (6) is determined from the condition

$$\lambda_k \omega^k(d) = -\tfrac{1}{2}\, d \ln \sigma.$$

Therefore, if H_0 does not fix one-dimensional directions, then all $\lambda_k = 0$, i.e., $\sigma = \text{const}$, and the space V_n is reducible in accordance with Theorem 1.

Remark 2. Since the transitive group G_r in (6) is an unmixed group of motions, under motions from G_r the function σ passes into $K\sigma$, and $a_{\alpha\beta}dx^\alpha dx^\beta$ into $\dfrac{1}{K} a_{\alpha\beta}dx^\alpha dx^\beta$. This means that the hypersurfaces $\sigma = \text{const}$ form in V_q a system of imprimitivity for the group of motions induced by G_r in V_q, while the metric $a_{\alpha\beta}dx^\alpha dx^\beta$ must be Euclidean.

Remark 3. The requirement that the group H_1 have no mutually permutable rotations is essential, for otherwise, as we have seen, the corresponding planes E_{n-q} may turn out to be nonholonomic, and there will no longer be a semireducible fibration of the space V_n. For example, the space V_4

$$ds^2 = e^{x^4}(dx^{1\,2} + dx^{2\,2}) + e^{2x^4}\left(dx^3 + x^1 dx^2 - x^2 dx^1\right)^2 + dx^{4\,2}$$

admits a transitive group of motions with translation operators

$$X_1 = p_3, \quad X_2 = p_2 + x^1 p_3, \quad X_3 = p_1 - x^2 p_3,$$

$$X_4 = -\frac{1}{2}x^1 p_1 - \frac{1}{2}x^2 p_2 - x^3 p_3 + p_4$$

and with the rotation operator $Y = x^1 p_2 - x^2 p_1$ relative to the point $(0,0,0,0)$. Here H_0 is the identity transformation in the plane $\{x^3, x^4\}$; H_1 is the group of rotations in the Euclidean plane $\{x^1, x^2\}$. Obviously, H_1 is irreducible, but there is no semireducible decomposition of V_4, in view of the fact that H_1 admits rotations permutable with it—in the present case it is permutable with itself. Analogous examples could be given for any n (for $n = 5$, see (4)).

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Note: Figure translations are in progress. See original paper for figures.