EXTENSION OF A GENERALIZED RANDOM PROCESS TO A COMPLETELY ADDITIVE MEASURE

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Abstract

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EXTENSION OF A GENERALIZED RANDOM PROCESS TO A COMPLETELY ADDITIVE MEASURE

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Every random process is defined by specifying consistent probability distributions for the values $\xi(t_1), \xi(t_2), \dots, \xi(t_k)$ of a random trajectory $\xi(t)$ at arbitrary times $t_1, t_2, \dots, t_k$. In this connection, naturally, two questions arise:

1. Can such consistent distributions (measures), given on cylindrical sets in the space of trajectories, be extended to a completely additive measure, already defined on all B-sets of some functional space?
2. What is, in fact, the stock of trajectories $\xi(t)$ realizing the given random process? In other words, in each case one is required effectively to single out as narrow as possible a set of functions possessing full measure.

The answer to the first question is given by a theorem of A. N. Kolmogorov (^{2}), which asserts that every random process (consistent distributions on cylindrical sets) can be extended to an additive measure in the space $\widetilde{R}$ of all functions $\varphi(t)$ on the line (taking finite values); the topology in $\widetilde{R}$ is generated by pointwise convergence of the functions $\varphi(t)$. The second question has been considered in numerous works as applied to one or another class of random processes (or to one or another random process). However, many practically interesting random phenomena cannot be naturally described within the framework of ordinary random processes (for example, "white noise" –the derivative of the Wiener process, the derivative of the Poisson process, etc.). In this connection I. M. Gel' fand (^{1}) and Ito (^{3}) independently introduced the concept of generalized random processes. For such processes the two questions mentioned above again arise:

1. Is it possible, in some space, to extend a generalized random process to an additive measure?
2. If so, what is the stock of elements of this space that has full measure?

In the present work the first of these questions is solved, and estimates are given that help, in many concrete cases, to answer the second.

In this work we follow the definition of a generalized random process given by

I. M. Gel’ fand.

Definition of a generalized random process. Let E be a certain real linear topological space, and let E' be the space conjugate to it*. For every finite set of linearly independent elements of E, $\varphi_1, \varphi_2, \dots, \varphi_n$, we specify a distribution probabilities of the values $F(\varphi_1), F(\varphi_2), \dots, F(\varphi_n)$ of the functional F. The set of functionals $\mathfrak{A}(F)$ satisfying, for some collection $\varphi_1, \dots, \varphi_n$, the equation

$$A\{F(\varphi_1), \dots, F(\varphi_n)\} \in R$$

(where A is a point of n-dimensional space with coordinates $F(\varphi_1), F(\varphi_2), \dots, F(\varphi_n)$, and R is a B-set in n-dimensional space), will be called a **cylindrical set** in E'. Thus, the probability distribution of the values $F(\varphi_1), F(\varphi_2), \dots, F(\varphi_n)$ generates a normalized measure, defined on cylindrical sets in E':

$$\mu_{\varphi_1 \dots \varphi_n}[\mathfrak{A}(F)].$$

Suppose that the following conditions are satisfied:

1. **Consistency:** if the cylindrical set $\mathfrak{A}(F)$ can be defined by means of two linearly independent collections of elements from E': $\varphi_1, \dots, \varphi_n$ and $\widetilde{\varphi}_1, \dots, \widetilde{\varphi}_s$, then the measure assigned to $\mathfrak{A}(F)$ is the same in both cases:

$$\mu_{\varphi_1 \dots \varphi_n}[\mathfrak{A}(F)] = \mu_{\widetilde{\varphi}_1 \dots \widetilde{\varphi}_s}[\mathfrak{A}(F)].$$

2. **Continuity:** for every $\varepsilon > 0$ there exists a neighborhood of zero U in the space E and a number $A > 0$ such that, for any $\varphi \in U$, the set $\mathfrak{A}(\varphi, A)$ of functionals F satisfying the condition $|F(\varphi)| > A$ has measure less than ε:

$$\mu[\mathfrak{A}(\varphi, A)] < \varepsilon.$$

A measure $\mu(\mathfrak{A})$, defined on cylindrical sets $\mathfrak{A}$ in E' and satisfying conditions 1, 2, is called a **generalized random process**.

Before formulating the main result, let us give the definition of an important class of linear topological spaces.

Definition of a nuclear space. Suppose that the topology in the linear space E is given by a countable collection of scalar products $(\varphi_1, \varphi_2)_n$, such that the following conditions are satisfied: 1) $(\varphi, \varphi)_{n+1} \geqslant (\varphi, \varphi)_n$; 2) the sum of the squares of the axes of the ellipsoid $(\varphi, \varphi)_{n+1} = 1$, measured in the n-th scalar product, converges. The space E with such a topology is called a **nuclear space**.

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* Recall that the space conjugate to the space E is the linear space whose elements are the continuous linear functionals defined on the space E.

We now formulate the main result of the paper.

Theorem 1. *Every generalized random process, defined in the space E' conjugate to a nuclear space E, extends to a completely additive measure, defined on the B-sets of the space E' (in the strong topology).*

In proving this theorem we shall rely on a lemma due to V. D. Erokhin.

Lemma 1. *In order that a generalized random process, defined in the space E' conjugate to a countably normed* space E, extend to a completely additive measure in E', it is necessary and sufficient that, for every $\varepsilon > 0$, there exist a sphere $U(\|F\|_n = S)$ in E' such that every cylindrical set lying outside this sphere have measure less than ε.*

We shall formulate two more preliminary lemmas.

Lemma 2. *Let an ellipsoid be given in Hilbert space,*

$$(Ax, x) = 1$$

(where A is a positive Hermitian operator), and a collection of mutually orthogonal vectors $\{e_i\}$ lying on the ellipsoid $(Ae_i, e_i) = 1$. Then

$$\sum_i \|e_i\|^2 < \operatorname{sp} A^{-1}.$$

Lemma 3. *Consider in n-dimensional space the quadratic form (Ax, x). The mean value of this form on the sphere $S(R)$ of radius R:*

$$\sum |x_i|^2 = R^2$$

is equal to

$$\langle Ax, x\rangle = \frac{R^2}{n} \operatorname{sp} A. \tag{1}$$

Lemma 4 (principal). *Suppose that in n-dimensional space there is defined* normalized measure $\mu(S)$. Suppose, further, that the ellipsoid

$$\sum \frac{x_i^2}{a_i^2} = 1 \tag{2}$$

has the property that, for any of its supporting planes, the measure of the outer half-space is less than ε. In that case the region exterior to the ball $T(R)$ of radius R has measure $\mu[CT(R)]$ less than $\beta(\varepsilon + H/R^2)$, where $H = \sum a_i^2$, and

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* A linear topological space is called countably normed if the topology in it is given by a countable collection of norms $\|\varphi\|_n$.

β is an absolute constant. (The ellipsoid (2) is entirely contained inside the ball $T(R)$.)

Proof. I. Consider the sphere $S(R)$ of radius R in n-dimensional space and the plane Γ at distance $\rho = R/\sqrt{n}$ from the origin. Then the normalized area of the cap cut off from the sphere $S(R)$ by the plane Γ is equal to

$$\alpha_n = C\int_{R/\sqrt{n}}^{R}\left(1-\frac{x^2}{R^2}\right)^{(n-2)/2}dx = C'\int_{1}^{\sqrt{n}}\left(1-\frac{\xi^2}{n}\right)^{(n-2)/2}d\xi,$$

where C' is determined from the condition

$$C'\int_{-\sqrt{n}}^{\sqrt{n}}\left(1-\frac{\xi^2}{n}\right)^{(n-2)/2}d\xi = 1.$$

We note that for $n \geq 2$, $\alpha_n \geq \alpha > 0$, where α is some absolute constant.

II. The distance of the supporting plane to the ellipsoid (2) from the origin is equal to

$$\rho(\omega) = \left(\sum a_i^2\omega_i^2\right)^{1/2},$$

where $\omega(\omega_1, \omega_2, \dots, \omega_n)$ is a unit vector perpendicular to this plane. The mean value of $\rho^2(\omega)$ over all directions ω is equal to

$$\langle\rho^2(\omega)\rangle = \frac{1}{n}\sum a_i^2 = \frac{H}{n}$$

(in the set of directions ω we assign the natural measure $\tau(\Omega)$, generated by the normalized area on the unit sphere $\sum \omega_i^2 = 1$).

Let Ω_1 be the set of those directions ω for which $\rho^2(\omega) > R^2/n$. The measure $\tau(\Omega_1)$ of such directions is, evidently, less than H/R^2: $\tau(\Omega_1) < H/R^2$. For the remaining directions ω, forming the set Ω_2, $\rho(\omega) < R/\sqrt{n}$. Draw, perpendicular to each direction ω, the plane $\Gamma(\omega)$ at distance $R/\sqrt{n}$ from the origin. Obviously, for $\omega \in \Omega_2$ the plane $\Gamma(\omega)$ lies farther from the origin than the corresponding supporting plane to the ellipsoid (2). Therefore, for such planes the measure of their outer half-spaces $L(\omega)$ is less than ε: $\mu(L(\omega)) < \varepsilon$.

III. Let us return to our measure $\mu(S)$ in n-dimensional space. Average this measure over all rotations U of the n-dimensional space, i.e., for each set S put

$$\mu_{\text{av}}(S) = \int \mu(US)\,dU,$$

where US is the image of the set S under the rotation U, and $\int \cdots dU$ is the invariant integral over the group of rotations U. We note that after such averaging the measure of any ball $T(R)$ (as well as of its complement $CT(R)$) will not change:

$$\mu_{\text{av}}[T(R)] = \mu[T(R)], \qquad \mu_{\text{av}}[CT(R)] = \mu[CT(R)].$$

Now consider some circular cone Λ with vertex at the origin and its intersection with the exterior of the ball $T(R)$: $\Lambda \cap CT(R)$.

Obviously, $\mu_{\text{av}}[\Lambda \cap CT(R)] = \alpha_n(\Lambda)\mu_{\text{av}}[CT(R)] = \alpha_n(\Lambda)\mu[CT(R)]$, where $\alpha_n(\Lambda)$ is the normalized area of the cap cut off by the cone Λ on the sphere $S(R)$. Let us note that if a plane cuts off on the sphere the same cap as the cone Λ, then its exterior half-space L entirely contains the intersection of the cone with the exterior of the ball: $L \supset \Lambda \cap CT(R)$. Hence it follows that $\mu_{\text{av}}(L) > \mu_{\text{av}}[\Lambda \cap CT(R)]$.

IV. Choose some plane Γ at distance $\rho = R/\sqrt{n}$ from the origin. As we have already seen, the cap cut off on the sphere $S(R)$ by this plane has normalized area greater than α.

Consider the averaged measure of the exterior half-space L for the plane Γ

$$\mu_{\text{av}}(L) = \int_{\Omega} \mu[L(\omega)]\, d\omega, \tag{3}$$

where $L(\omega)$ is the exterior half-space for the plane $\Gamma(\omega)$ at distance $R/\sqrt{n}$ from the origin and perpendicular to the direction ω.

The integral (3) over all directions ω splits into the sum

$$\mu_{\text{av}}(L) = \int_{\Omega_1} \mu[L(\omega)]\, d\omega + \int_{\Omega_2} \mu[L(\omega)]\, d\omega,$$

where Ω_1 is the set of those directions for which $\rho(\omega) > R/\sqrt{n}$, and Ω_2 the directions for which $\rho(\omega) < R/\sqrt{n}$. As we have seen, $\mu[L(\omega)] < \varepsilon$ if $\omega \in \Omega_2$, and $\tau(\Omega_2) < H/R^2$, while always $\mu[L(\omega)] \leqslant 1$. Hence we obtain $\mu_{\text{av}}(L) \leqslant \varepsilon + H/R^2$.

Now take a cone Λ which cuts off on the sphere the same cap as the plane Γ. We have $\mu_{\text{av}}[\Lambda \cap CT(R)] \leqslant \mu_{\text{av}}(L) \leqslant \varepsilon + H/R^2$ and $\mu_{\text{av}}[\Lambda \cap CT(R)] = \alpha_n \mu[CT(R)] \geqslant \alpha\mu[CT(R)]$. Finally we obtain

$$\mu[CT(R)] \leqslant \beta\left(\varepsilon + \frac{H}{R^2}\right), \qquad \beta = \frac{1}{\alpha}.$$

The lemma is proved.

Proof of the theorem. According to Lemma 1 of V. D. Erokhin, for every $\mu > 0$ it is necessary to indicate such a ball $T(R)$ in E' that the measure $\mu(S)$ of any cylindrical set lying outside this ball is less than μ. Choose $\varepsilon = \mu/2\beta$ and construct a ball of radius ρ in some k-th norm so that the measure of the exterior half-space for any supporting plane of this ball is less than ε (the continuity condition). Then, in the following $(k+1)$-st norm, choose a ball of such radius R that $H/\beta R^2 < \mu/2$, where H is the sum of the semiaxes of the ellipsoid $(\varphi, \varphi)_{k+1} = 1/\rho^2$ in the space E. Obviously, the ball of radius R will satisfy the requirement of Lemma 1. The theorem is proved.

Let us note that, for the extendability of any generalized random process to an additive measure in E', the requirement that E be a nuclear space is in some sense necessary, namely:

Theorem 2. *If in the space E', conjugate to some countably normed space E, every generalized random process can be extended to a completely additive measure, then the space E is nuclear (isomorphic to a nuclear one).*

In conclusion I express my gratitude to I. M. Gel' fand, who posed the problem of extending a measure and also first stated the assertion that this problem has a positive solution for nuclear spaces. I also thank V. D. Erokhin, whose lemma was essential for the proof of the main theorem.

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Note: Figure translations are in progress. See original paper for figures.

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