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Abstract

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ON UNIVERSAL CLASSES OF ALGEBRAS

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We shall consider algebras of a single type with a finite number of fundamental operations. Following A. I. Mal'cev ([1]), we shall call a class of algebras **abstract** if, together with each of its algebras, it contains all algebras isomorphic to it. In all that follows we shall consider abstract classes of algebras. A class of algebras is called **universal** if it is defined by a system (finite or infinite) of axioms of the form

$$(x_1)\dots(x_n)\{F\}, \tag{1}$$

where F is a formula containing no quantifiers. Every system of axioms of the form (1) is equivalent to some system of axioms of the form

$$(x_1)\dots(x_n)\{f_1 = g \vee \dots f_l = g_l \vee \sim h_1 = q_1 \vee \dots \vee \sim h_m = q_m\}, \tag{2}$$

where $f_1, \dots, q_m$ are polynomials in $x_1, \dots, x_n$.

The **universal closure** $U(K)$ of a class K will mean the smallest universal class containing K. The operation of universal closure defines, on the set of all (single-type) algebras, a topological structure UC_δ. We shall say that a class $L \subset K$ is **universal in** K, or **universal relative to** K, if L is a closed set in the structure $UC_\delta(K)$ induced in K by the structure UC_δ.

In the present note structural properties are established which characterize universal and relatively universal classes of algebras. We note that in many of the propositions given here, the assumption that algebras, and not models in general, are being considered is not essential.

The factor algebra of an algebra $\mathfrak{A}$, determined by a congruence relation ε, will be denoted by $\mathfrak{A}/\varepsilon$.

Let K be some class of algebras, and $\mathfrak{A}$ an arbitrary algebra. A congruence relation ε on $\mathfrak{A}$ is called a K**-congruence** if $\mathfrak{A}/\varepsilon \in K$.

Theorem 1. *Let K be a universal class, and let $\mathfrak{A}$ be an arbitrary algebra. If the K-congruences on $\mathfrak{A}$ form a subsemilattice (a lattice of all congruences on $\mathfrak{A}$), then this subsemilattice is complete.*

We shall call a class K **locally definable** if it has the following property: an algebra $\mathfrak{A}$ belongs to K if and only if every finitely generated subalgebra of it belongs to K. We note that the term "locally definable class," borrowed from the article of A. I. Mal' cev (^{2}), is used here in a broader sense. The example of the class of all algebras which do not contain free algebras as their subalgebras shows that local definability of a class in the sense indicated above does not imply local definability in the sense of Mal' cev (equivalent to universality of the class).

Let M be some set, and $\{M_i\}$ a sequence of its subsets. Following Hausdorff (^{3}), the set $\lim M_i$ of all such $a \in M$,

that a belongs to all M_i, beginning with some i_a, will be called the lower limit of the sequence $\{M_i\}$.

It is easily established that

$$\underline{\lim}\, M_i = \bigcup_{i=1}^{\infty} \bigcap_{j=i}^{\infty} M_j.$$

If the lower limit of the sequence $\{M_i\}$ is equal to the lower limit of every one of its subsequences, then $\{M_i\}$ is called a convergent sequence, and $\lim M_i$ is then denoted by $\lim M_i$.

It is obvious that for any algebra $\mathfrak{A}$ the lower limit of every sequence of congruences on $\mathfrak{A}$ (regarded as subsets of the set $\mathfrak{A} \times \mathfrak{A}$) is a congruence.

We shall say that a set $\{\varepsilon\}$ of congruences on $\mathfrak{A}$ is closed if $\{\varepsilon\} = \Lambda$, where Λ is the empty set, or if for every convergent sequence $\{\varepsilon_i\} \subset \{\varepsilon\}$ one has $\lim \varepsilon_i \in \{\varepsilon\}$.

We shall say that a class of algebras K is closed if, for every algebra $\mathfrak{A}$, the set of K-congruences on $\mathfrak{A}$ is closed.

A sequence $\{\mathfrak{A}_i\}$ of subalgebras of an algebra $\mathfrak{A}$ is called a chain if, for any $\mathfrak{A}_i$ and $\mathfrak{A}_j$, $\mathfrak{A}_i \subset \mathfrak{A}_j$ or $\mathfrak{A}_j \subset \mathfrak{A}_i$.

It is easy to show that if K is a closed class, then for every chain $\{\mathfrak{A}_i\} \subset K$, $\bigcup \mathfrak{A}_i \in K$, and also $\bigcap \mathfrak{A}_i = K$ or $\bigcap \mathfrak{A}_i = \Lambda$.

Hence, in particular, it follows that a closed class is a class with natural generations and, consequently, if it is axiomatizable, then it is axiomatizable only by means of axioms of Skolem form (see (4), theorem 2).

Apparently the following hypothesis is true, although the author has not yet succeeded in proving it in full generality:

An axiomatizable class is closed if and only if it is universal.

Lemma 1. *Every universal class is closed.*

Let Σ be a system of axioms of the form (2) defining the class K; let $\mathfrak{A}$ be an arbitrary algebra; let $\{\varepsilon_i\}$ be a convergent sequence of K-congruences on $\mathfrak{A}$. Suppose that $\mathfrak{A}/\lim\varepsilon_i$ does not satisfy some axiom (2) from Σ, i.e. $\mathfrak{A}/\lim\varepsilon_i$ satisfies the axiom

$$(\exists x_1)\dots(\exists x_n)\{\sim f_1 = g_1 \vee \dots \wedge \sim f_l = g_l \wedge h_1 = q_1 \wedge \dots h_m = q_m\}.$$

Consequently, in $\mathfrak{A}$ there are elements $x_1^0,\dots,x_n^0$ such that:

a) there exist $i_1,\dots,i_l$ such that, for every $i > i_1$,

$$(f_1(x_1^0,\dots,x_n^0),\ g_1(x_1^0,\dots,x_n^0)) \notin \varepsilon_i,\dots;$$

for every $i > i_l$,

$$(f_l(x_1^0,\dots,x_n^0),\ g_l(x_1^0,\dots,x_n^0)) \notin \varepsilon_i;$$

b) there exist $j_1,\dots,j_m$ such that, for every $j > j_1$,

$$(h_1(x_1^0,\dots,x_n^0),\ q_1(x_1^0,\dots,x_n^0)) \in \varepsilon_j,\dots;$$

for every $j > j_m$,

$$(h_m(x_1^0,\dots,x_n^0),\ q_m(x_1^0,\dots,x_n^0)) \in \varepsilon_j.$$

Let $i_0 = \max(i_1,\dots,j_m)$. Then, obviously, $\mathfrak{A}/\varepsilon_{i_0} \notin K$, which contradicts the assumption.

Let $\mathfrak{A}$ be an arbitrary algebra, and M some set of its elements. By $(\mathfrak{A}, M, i)$ we shall denote the submodel of the algebra $\mathfrak{A}$ whose elements are all possible values of polynomials of order not higher than the i-th, with all possible values of the arguments from M.

The class of all subalgebras of all algebras from K will be denoted by $S(K)$, and the least closed class containing K by $\overline{K}$.

Lemma 2. *If $K = \overline{K} = S(K)$, then every finitely generated algebra $\mathfrak{A}$ from $U(K)$ belongs to K.*

Indeed, let $M = \{a_1,\dots,a_n\}$ be a system of generators of the algebra $\mathfrak{A} \in U(K)$. It is easy to show that, for every i, there exists an algebra $\mathfrak{A}_i' \in K$ into which $(\mathfrak{A}, M, i)$ is isomorphically embeddable (see (5), theorem 1.1). Consequently, for every i there exists an algebra $\mathfrak{A}_i \in K$ such that

$M_i = \{a_1^i,\dots,a_n^i\}$ is a system of its generators, and its submodel $(\mathfrak{A}_i, M_i, i)$ is isomorphic to $(\mathfrak{A}, M, i)$. Let $\mathfrak{U}$ be the free algebra with generators $U_1,\dots,U_n$; let ε_i be the congruence on $\mathfrak{U}$ corresponding to the homomorphic mapping of

$\mathfrak{U}$ onto $\mathfrak{A}_i$ such that U_1 is mapped to $a_1^i, \ldots, U_n$ to a_n^i $(i = 1, 2, \ldots)$. It is easy to show that the sequence of K-congruences $\{\varepsilon_i\}$ converges and $\mathfrak{U}/\lim \varepsilon_i \simeq \mathfrak{A}$. Consequently, $\mathfrak{A} \in K$.

From Lemmas 1 and 2 it follows easily:

Theorem 2. *A class of algebras is universal if and only if it is locally definable and closed.*

It is of interest to determine whether, in Theorem 2, the condition of local definability of the class can be replaced by the condition that the class is closed with respect to the formation of subalgebras.

We shall say that the class $L = K$ has property σ in K if there exists a class M having property σ such that $L = K \cap M$.

Theorem 3. *Let K_ω be the class of all finite and countable algebras. A class $L \subset K_\omega$ is universal in K_ω if and only if L is closed in K_ω and $L = S(L)$.*

The proof follows from Lemma 2 and from the fact that for every chain $\{\mathfrak{A}_i\}$ of algebras of a closed class L, $\bigcup \mathfrak{A}_i \in L$.

Theorem 4. *A class $L \subset K = S(K)$ is universal in K if and only if it is locally definable in K and closed in K.*

It is easy to show that a universal class of algebras is closed with respect to direct products if and only if, for every algebra $\mathfrak{A}$, the set of K-congruences on $\mathfrak{A}$ is a lower subsemilattice or is empty. Hence, using Theorem 2, it is easy to derive the following two theorems.

Theorem 5. *A class K closed with respect to direct products is universal if and only if it is locally definable and has the following property: for every algebra $\mathfrak{A}$ and every chain $\{\varepsilon_i\}$ of K-congruences on $\mathfrak{A}$, $\mathfrak{A}/\bigcup \varepsilon_i \in K$.*

Theorem 6. *Let a class K be locally definable and have the following property: for every algebra $\mathfrak{A}$ and every sequence $\{\varepsilon_i\}$ of K-congruences on $\mathfrak{A}$, $\lim \varepsilon_i$ is a K-congruence. Then and only then is it universal and closed with respect to direct products.*

A universal class closed with respect to direct products is defined by such a system of axioms Σ that every axiom of Σ has one of the following two forms:

$$(x_1) \ldots (x_n)\{\sim f_1 = g_1 \vee \ldots \vee \sim f_l = g_l\}; \tag{3}$$

$$(x_1) \ldots (x_m)\{h_1 = q_1 \wedge \ldots \wedge h_l = q_l \rightarrow h_{l+1} = q_{l+1}\}. \tag{4}$$

Following A. I. Mal' cev ([2]), a class defined by a system of axioms of the form (4) will be called a **quasiprimitive class**.

An algebra with a unit element will be called a 0-algebra. It is easy to see that a 0-algebra satisfies no axiom of the form (3) and satisfies every axiom of the form (4). Hence it follows:

Theorem 7. *A class of algebras is quasiprimitive if and only if it satisfies the conditions of Theorem 6 and contains a* 0*-algebra.*

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