ON SOME EXACT SOLUTIONS OF THE EQUATIONS OF GAS DYNAMICS

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Abstract

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In theoretical gas dynamics the following question is of interest. Is there possible a motion of a perfect, ideal, non-heat-conducting gas with a shock wave that has the property that behind the shock wave there exists a region in which the speed of sound is nonzero, but if disturbances of the motion are produced in it, these disturbances will never catch up with the shock wave; in particular, is it possible to create in this region a second shock wave propagating after the first, but likewise never catching up with it? In the work of A. S. Kompaneets ([1]) an attempt was made to answer this question in the affirmative; however, the two shock waves present in A. S. Kompaneets' solution in fact move through the gas in opposite directions, and therefore this solution contains no answer to the question posed. In other works (see, for example, ([2])) this question was studied in one or another approximate formulation; however, owing to such a formulation, no rigorous answer to the question posed can be extracted from the results obtained there. In the present note an exact answer to this question is given by considering some exact solutions of the equations of nonstationary gas dynamics.

Let us consider the following initial problem:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0, \qquad \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0,$$
$$\frac{\partial}{\partial t}\left(\frac{p}{\rho^\gamma}\right) + u\frac{\partial}{\partial x}\left(\frac{p}{\rho^\gamma}\right) = 0; \tag{1}$$

$$p(x,0) = p_0 e^{x/a}, \qquad \rho(x,0) = \rho_0 e^{x/b}, \qquad u(x,0) = u_0 e^{x/c}. \tag{2}$$

If the constants a, b, c are connected by the relation $c = 2ab/(b-a)$, then ([3]) the motion of the gas will be a limiting self-similar one:

$$u(x,t) = \frac{a}{t}V(\lambda), \qquad p(x,t) = p_0\frac{a^2}{t^2}e^{x/b}P(\lambda), \qquad \rho(x,t) = \rho_0 e^{x/b}R(\lambda); \tag{3}$$

$$\lambda = \sqrt{\frac{p_0}{\rho_0}}\,\frac{t}{a}\exp\left(\frac{b-a}{2ab}x\right), \tag{4}$$

where $V(\lambda), R(\lambda), P(\lambda) = R(\lambda)z(\lambda)/\gamma$ are a solution of the equations

$$\frac{dz}{dV} = z\,\frac{\left(2-\frac{\gamma-1}{\gamma}n\right)z + (\gamma-1)V(V+n) - 2(V+n)^2}{(V+n)\left[\frac{n}{\gamma}z - V(V+n)\right]}; \tag{5}$$

$$\frac{d\ln\lambda}{dV} = \frac{z-(V+n)^2}{n\left[\frac{n}{\gamma}z - V(V+n)\right]}. \tag{6}$$

$$\frac{V+n}{n}\,\frac{d\ln R}{d\ln\lambda} = -\frac{\frac{n}{\gamma}z - V(V+n)}{z-(V+n)^2}, \qquad n = \frac{2b}{b-a}. \tag{7}$$

The integral curves of equation (5), which for small λ give the solution of the Cauchy problem, have the asymptotic representation [3]

$$z = \frac{\gamma}{M^2}V^2 + \ldots\ (M \neq 0); \qquad z = -\gamma V + \ldots\ (M = 0); \qquad M = u_0\sqrt{\frac{\rho_0}{p_0}}. \tag{8}$$

To complete the construction of the solution of the Cauchy problem, these integral curves must be carried to the points at which $\lambda = \infty$, or $\lambda = \lambda_{\text{gr}}$, where λ_{gr} corresponds to the moving gas boundary, if it arises in the course of the motion. The field of integral curves of equation (5) for the case $n < 0$ is shown in Fig. 1. The arrows indicate the direction of increasing λ. In the case $n > 0$,[3] the solution of the problem posed, continued to all times $t > 0$, either does not exist at all or is nonunique; therefore this case is of no interest to us here. In [3] it is shown that for all values of the constant $M < M_0$, where M_0 corresponds to the separatrix OA, the solution of the problem is continuous and is given by integral curves issuing to the left of OA and tending to infinity as $V \to -\infty$. For curves issuing to the right of OA ($M > M_0$), the solution is constructed as follows: from these curves one must make a jump to the other separatrix passing through A (it can be proved [3] that for each of the curves mentioned this jump can be made in a unique way, namely, the jump must be made from points lying on the dash-dotted curve which is the image of the second separatrix obtained under the mapping (9)), and along it, passing through A, go off to infinity as $V \to -\infty$. The jump is made in accordance with the conditions on a shock wave [3,4]:

Fig. 1

Fig. 1

Figure 1: Fig. 1

$$V_2 + n = (V_1 + n)\left[1 + \frac{2}{\gamma+1}\frac{z_1 - (V_1+n)^2}{(V_1+n)^2}\right], \qquad R_2 = R_1\frac{V_1+n}{V_2+n},$$
$$z_2 = \left(\frac{\gamma-1}{\gamma+1}\right)^2\left[1 + \frac{2}{\gamma-1}\frac{z_1}{(V_1+n)^2}\right]\left[\frac{2\gamma}{\gamma-1}(V_1+n)^2 - z_1\right]. \tag{9}$$

These relations map, in a one-to-one manner, the points of the region

$$0 \le z \le (V+n)^2$$

to points of the region

$$(V+n)^2 \le z \le \frac{2\gamma}{\gamma-1}(V+n)^2.$$

Let us consider the solutions corresponding to $M > M_0$. For them, the part of the curve from the point O to the point of discontinuity corresponds in physical space

region ahead of the shock wave. The second piece of the solution curve, i.e. the piece of the separatrix, corresponds in physical space to the region behind the shock wave. At the point O, $\lambda = 0$, i.e., for fixed t it corresponds to the point $x = +\infty$ (without loss of generality one may assume $a > 0$). At the discontinuity $\lambda = \lambda_0 > 0$, at the point A, $\lambda = \lambda_* > \lambda_0$, and at the point $V = -\infty$, $\lambda = \infty$, which corresponds to $x = -\infty$. $\lambda(x,t) = \lambda_0$ is the law of motion of the shock wave; $\lambda(x,t) = \lambda_*$ is the law of motion of a certain surface dividing the region behind the shock wave into two parts. It is easy to show that in the region $\lambda_0 < \lambda < \lambda_*$ the speed of propagation of sound disturbances behind the shock wave is greater than the speed of propagation of the states $\lambda = \text{const}$, whereas in the region $\lambda_* < \lambda < \infty$ it is, conversely, less than the speed of the states $\lambda = \text{const}$.

It follows immediately from this that small disturbances produced in the region $\lambda_* < \lambda < \infty$ can never pass into the region $\lambda_0 < \lambda < \lambda_*$ and, in particular, can never catch up with the shock wave. Thus the region $\lambda_0 < \lambda < \lambda_*$ is insensitive to small disturbances produced in the rest of the space behind the shock wave.

If we now supplement the original formulation of the problem by the assumption that from the point $x = -\infty$ the gas is displaced by a piston moving according to the law $\lambda(x,t) = \lambda_1$ (λ_1 is a prescribed number), then the self-similarity will not be violated, and the solution can be constructed using the same field of integral curves (Fig. 1). In this case the curves issuing from O can no longer be sent to infinity as $V \to -\infty$, for now the region of variation of λ is bounded: $0 < \lambda < \lambda_1$. The solution is constructed as follows. From the given integral curve issuing from O (with M prescribed), one must make a jump to the curves going to the point $V = -n,\ z = \infty$. It can be shown that at this point λ assumes a finite value and the velocity of the gas particles here coincides with the velocity of the piston. However, depending on where on the given curve we make the jump, we shall arrive at different curves and reach the point $V = -n,\ z = \infty$ with different values of λ. In order to obtain the prescribed value $\lambda = \lambda_1$, this jump must be made from a quite definite point of the original integral curve. To show that for any values M $(-\infty < M < +\infty)$ and λ_1 $(0 < \lambda_1 < \infty)$ such a jump can be made, let us consider the images of all curves issuing from O and situated in the region $0 \leq z \leq (V + n)^2$, obtained under the mapping (9). The qualitative picture of the distribution of these images is shown in Fig. 1 (dashed curves); O_1 is the image of O. It can be shown that if the point on the given curve from which the jump is made moves along the curve from the point $V = -\infty$ toward O, then its image moves along the image of the curve toward O_1, while the values λ_1 obtained at the point $V = -n,\ z = \infty$ vary from $\lambda_1 = \infty$ to $\lambda_1 = 0$, thereby exhausting all possible prescribed laws of motion of the piston. This holds for all initial integral curves, i.e. for all M, as required.

We further note that for $M < M_0$ the motion of the gas in the presence of the piston occurs with one shock wave, whereas in the absence of the piston this motion is continuous. For the case $M > M_0$, however, the motion for sufficiently large values of λ_1 occurs with two shock waves, whereas in the absence of the piston it occurred with one shock wave. If λ_1 is decreased, i.e. if the point on the piece of the separatrix lying to the left of A, from which the jump is made, is moved toward A, then the image of this point will also move toward A, and they will reach it simultaneously at some value $\bar{\lambda}_1$. The second shock wave then disappears. For $\lambda_1 < \bar{\lambda}_1$ the jump must be made from points of the corresponding curve issuing from O, and as $\lambda_1 \to 0$ the point from which the jump is made will approach O, and its image will approach O_1. Thus, for every $M > M_0$ there exists a $\bar{\lambda}_1$ such that, if $\lambda_1 > \bar{\lambda}_1$, the solution contains two shock waves traveling through the gas one behind the other, the rear one never catching up with the front one.

If, however, $\lambda_1 < \bar{\lambda}_1$, then the motion occurs with a single shock wave. In these solutions the motion ahead of the second shock wave is determined only by the initial conditions, while the disturbances generated by the piston are confined to the region between the piston and this shock wave. If λ_1 in the law of motion of the piston is decreased, then the second shock wave, being, evidently, in the region $\lambda > \lambda_*$, will approach ever closer to the boundary of this region and, for $\lambda_1 = \bar{\lambda}_1$, will reach this boundary. Any further decrease of λ_1, however small, immediately leads to a restructuring of the entire flow everywhere behind the first shock wave and changes the law of motion of this wave. In the limit as $\lambda_1 \to 0$, the shock wave, the piston, and the entire gas are all instantaneously carried to the point $x = +\infty$.

Thus, exact solutions have been considered in which, in the region $\lambda > \lambda_*$, there is a disturbance that is not small but finite—a shock wave—which likewise does not overtake the first shock wave. It is clear that the motion in this region can be disturbed not only by introducing a piston that does not violate the self-similarity of the problem, but also in many other ways that violate the self-similarity of the motion as a whole; moreover, this can be done so that the disturbances, which may also be shock waves, do not penetrate into the region $\lambda < \lambda_*$, i.e., do not overtake the leading shock wave and the adjacent region of flow.

In conclusion we point out other classes of exact solutions of the equations of gas dynamics possessing the properties considered here and contained among the ordinary (not limiting) self-similar one-dimensional unsteady motions of a gas. Thus, for example, applying the arguments given above to the consideration of fields of integral curves given in [4] on p. 199 (Figs. 32 and 33), one can establish that the solutions of the Cauchy problems corresponding to these fields of integral curves (in [5] it is shown that to any field of integral curves for an arbitrary self-similar motion of a gas there corresponds a certain self-similar Cauchy problem) possess the required property.

Let us also note that, in the case of the limiting self-similar problem considered above, the distance between the shock waves remains constant with time, whereas in the case of an ordinary self-similar problem it grows as t^α.

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Note: Figure translations are in progress. See original paper for figures.