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Abstract

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MATHEMATICS

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**ON THE AIZERMAN PROBLEM FOR THE
CASE OF A SYSTEM OF THREE DIFFERENTIAL
EQUATIONS**

(Presented by Academician V. I. Smirnov on 21 III 1958)

Consider a system of three differential equations of Aizerman type ⁽¹⁾

$$\begin{aligned} dx/dt &= f_1(x) + a_{12}y + a_{13}z, \\ dy/dt &= a_{21}x + a_{22}y + a_{23}z, \\ dz/dt &= a_{31}x + a_{32}y + a_{33}z. \end{aligned} \tag{1}$$

This system, by a linear transformation of the desired variables and of the function $f_1(x)$, is reduced to the form

$$dx/dt = y + f(x), \quad dy/dt = z + ax + bf(x), \quad dz/dt = cx + df(x). \tag{2}$$

With respect to the function $f(x)$ we shall assume that it satisfies the condition of uniqueness of solutions of system (2) and the generalized Hurwitz conditions:

$$f(0) = 0, \quad f(x)/x < 0, \quad d(f(x)/x) + c < 0,$$

$$bf^2(x)/x^2 + (a + d)f(x)/x + c > 0 \quad \text{for } x \neq 0. \tag{3}$$

In the present paper, conditions are imposed on the parameters a , b , c , and d of system (2) which are necessary and sufficient for the zero solution of this system to be globally stable for all nonlinearities $f(x)$ satisfying the generalized Hurwitz conditions (3). In cases where the parameters of system (2) do not satisfy these conditions, such nonlinearities $f(x)$, subject to conditions (3), are indicated for which the zero solution of system (2) is not globally stable. The

case $b = 0$ has been studied exhaustively from this point of view in ⁽⁵⁾; therefore in what follows we shall assume that $b \neq 0$.

Put

$$A = \frac{-(a+d) - \sqrt{(a+d)^2 - 4bc}}{2b}, \quad B = \frac{-(a+d) + \sqrt{(a+d)^2 - 4bc}}{2b}.$$

Let us first consider the cases where the inequalities

$$b > 0, \quad d > 0, \quad (a+d)^2 - 4bc < 0 \quad (4)$$

hold, or

$$b > 0, \quad d > 0, \quad (a+d)^2 - 4bc \geq 0, \quad -c/d < \min\{A, 0\}. \quad (5)$$

In both these cases the generalized Hurwitz conditions (3), as is easy to verify, reduce to the inequality

$$f(x)/x < -c/d \quad (6)$$

and by simple transformations of the variables and of the function $f(x)$, system (2) is reduced to the system of so-called indirect control, completely studied in ⁽³⁾.

In the present paper the following result is obtained: in the cases when inequalities (4) or (5) are satisfied, the zero solution of system (2) is stable in the large for any nonlinearity $f(x)$ satisfying condition (6).

Let us turn to the case when the relations

$$b > 0, \quad d > 0, \quad (a+d)^2 - 4bc \geq 0, \quad A = -c/d = 0. \quad (7)$$

are satisfied. In this case the generalized Hurwitz condition is written in the form $f(x)/x < 0$. (This case was studied in detail by A. P. Tuzov ⁽⁴⁾.)

Using the results of the present paper, it is easy to prove that if relations (7) are satisfied, then the zero solution of system (2) is stable in the large for any $f(x)$ satisfying the condition $f(x)/x < 0$.

Suppose, further, that the relations

$$b > 0, \quad d > 0, \quad (a+d)^2 - 4bc \geq 0, \quad A = -c/d < 0 \quad (8)$$

are satisfied.

This case was investigated in ⁽⁶⁾, where it was shown that the zero solution of system (2) is stable in the large not for every $f(x)$ satisfying the generalized Hurwitz condition $f(x)/x < A$.

We shall now consider the case when the inequalities

$$b < 0, \quad d > 0, \quad B < -c/d \leq \min\{A, -c/d\}. \quad (9)$$

are satisfied. In this case conditions (3) take the form

$$B < f(x)/x < -c/d. \quad (10)$$

Theorem 1. *If inequalities (9) are satisfied, then the zero solution of system (2) is stable in the large for any $f(x)$ subject to condition (10).*

In the proof of this and all subsequent theorems on stability of motion in the large, a Lyapunov function of the form “the integral of the nonlinearity plus a quadratic form in the coordinates of the phase space” ⁽⁵⁾, and the general theorem formulated in ⁽⁷⁾, are used.

We pass to the case when the inequalities

$$b < 0, \quad d > 0, \quad A < \min\{0, -c/d\}. \quad (11)$$

are satisfied. The Hurwitz conditions (3) in this case take the form

$$B < f(x)/x < A. \quad (12)$$

As the following theorem shows, in this case the zero solution of system (2) is stable in the large not for all $f(x)$ satisfying inequalities (12).

Theorem 2. *Let inequalities (11) and the conditions*

$$Bx < f(x) \leq hx \quad \text{for } 0 < x \leq x_1, \quad (13)$$

$$0 < Ax - f(x) < \lambda \quad \text{for } x \geq x_1, \quad (14)$$

be satisfied, where the number h is subject to the inequality

$$h < \frac{(A^2 - a - bA)(d + a + bA)}{A(d + a + bA) + b(a + bA)}, \quad (15)$$

and the numbers λ , x_1 , $x_2 - x_1$ are positive and sufficiently small. Suppose, moreover, that $f(x) = -f(-x)$. Then system (2) has solutions that do not tend to the origin as $t \rightarrow +\infty$.

Consider the case in which the conditions

$$b < 0, \quad d = 0, \quad A < 0. \quad (16)$$

are satisfied. In this case the generalized Hurwitz conditions take the form

$$B < f(x)/x < A. \quad (17)$$

Theorem 3. Let inequalities (16) and conditions (13)–(14) be satisfied, in which the number h is subject to the inequality

$$h < \frac{A^2 - a - bA}{A + b}, \quad (18)$$

and the numbers λ , x_1 , $x_2 - x_1$ are positive and sufficiently small. Suppose, moreover, that $f(x) = -f(-x)$. Then system (2) has solutions which do not tend to the origin as $t \rightarrow +\infty$.

Let us turn to the case

$$b < 0, \quad d < 0, \quad -c/d < B < A < 0. \quad (19)$$

The generalized Hurwitz conditions (3) have the form (17), and the following theorem is valid.

Theorem 4. Let inequalities (19) and conditions (13), (14), (15) be satisfied; suppose that $f(x) = -f(-x)$. Then system (2) has solutions which do not tend to $(0, 0, 0)$ as $t \rightarrow +\infty$.

Suppose that the inequalities

$$b < 0, \quad d < 0, \quad B \leq -c/d < A < 0 \quad (20)$$

are satisfied. In this case conditions (3) are written in the form

$$-c/d < f(x)/x < A. \quad (21)$$

The following two theorems analyze this case.

Theorem 5. If inequalities (20) are satisfied and $A^2 + Ab + d \leq 0$, then the zero solution of system (2) is globally stable for any nonlinearity $f(x)$ satisfying the generalized Hurwitz conditions (21).

Theorem 6. Suppose that inequalities (20) are satisfied and $A^2 + Ab + d > 0$. Let the condition

$$-cx/d < f(x) \leq hx \quad \text{for } 0 < x \leq x_1 \quad (22)$$

and condition (14) be satisfied, where the number h is subject to inequality (15), and the numbers λ , x_1 , $x_2 - x_1$ are positive and sufficiently small. Suppose, moreover, that $f(x) = -f(-x)$. Then system (2) has solutions which do not tend to $(0, 0, 0)$ as $t \rightarrow \infty$.

Consider the case

$$b > 0, \quad d = 0, \quad A < 0. \quad (23)$$

The generalized Hurwitz conditions are written in the form $f(x)/x < A$. In this case, as in the preceding one, two possibilities may occur.

Theorem 7. If inequalities (23) are satisfied and $A + b \geq 0$, then the zero solution of system (2) is globally stable for any $f(x)$ satisfying the condition $f(x)/x < A$.

Theorem 8. Let inequalities (23) be satisfied and $A + b < 0$. Let the condition

$$f(x) \leq hx \quad \text{for } 0 < x \leq x_1 \quad (24)$$

and condition (14) be satisfied, where the number h is subject to inequality (18), and the numbers λ , x_1 , $x_2 - x_1$ are positive and sufficiently small. Suppose that $f(x) = -f(-x)$. Then the zero solution of system (2) is not globally stable.

Assume that the inequalities

$$b > 0, \quad d < 0, \quad -c/d < A < 0 \quad (25)$$

hold. The Hurwitz conditions in this case have the form (21).

Theorem 9. Let inequalities (25) be satisfied and $A^2 + Ab + d \leq 0$. Then the zero solution of system (2) is globally stable for any $f(x)$ satisfying condition (21).

Theorem 10. Suppose that inequalities (25) and all the conditions of Theorem 6, except inequalities (20), are satisfied. Then system (2) has solutions that do not tend to the origin of coordinates as $t \rightarrow +\infty$.

Finally, consider the case when

$$b > 0, \quad d > 0, \quad (a + d)^2 - 4bc \geq 0. \quad (26)$$

As follows from what was said above, of these cases only two remain to be considered.

$$B < -c/d < 0, \quad (27)$$

then conditions (3) take the form

$$B < f(x)/x < -c/d, \quad (28)$$

$$A < \min\{0, -c/d\} \quad (29)$$

and the generalized Hurwitz conditions are written in the form $f(x)/x < A$.

Theorem 11. If inequalities (26) and (27) are satisfied, then the zero solution of system (2) is globally stable under conditions (28).

Theorem 12. If inequalities (26), (29), and $b^2A + ab - c < 0$ are satisfied, then the zero solution of system (2) is globally stable for any nonlinearity $f(x)$ satisfying the generalized Hurwitz condition $f(x)/x < A$.

Theorem 13. Suppose that inequalities (26), (29), and $b^2A + ab - c > 0$ are satisfied, as well as conditions (24) and (14), in which the number h is subject to the inequality

$$h < \frac{bc - a^2 - ad - A(ab + c)}{b^2A + ab - c}, \quad (30)$$

and the numbers λ , x_1 , $x_2 - x_1$ are positive and sufficiently small. Let $f(x) = -f(-x)$. Then system (2) has solutions that do not tend to the origin of coordinates as $t \rightarrow +\infty$.

The theorems formulated also give, as is not difficult to verify, necessary and sufficient conditions for the stability of the zero solution of system (2) for arbitrary nonlinearities $f(x)$ satisfying the generalized Hurwitz conditions (3).

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