Reports of the Academy of Sciences of the USSR

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1958

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1958. Volume 119, No. 5

MATHEMATICS

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SOME TYPES OF LOCAL NOMOGRAMS

(Presented by Academician A. N. Kolmogorov on 30 XI 1957)

In papers ([1,2]) it was shown that any function $z = f(x, y)$ sufficiently smooth at the point x_0, y_0, such that $f'_x(x_0, y_0) \neq 0$, $f'_y(x_0, y_0) \neq 0$, is nomographable in a neighborhood of this point with accuracy up to small quantities of the 6th order. In this case the quantities $\alpha_1, \alpha_2, \beta_2, \gamma_3$—the coefficients in the rows of the Massau determinant effecting this nomographing—remain arbitrary.

In paper ([3]) it was proved that, for $f'_x(x_0, y_0) \neq 0$, $f'_y(x_0, y_0) \neq 0$, $P(x_0, y_0) \neq 0$, where $P(x, y)$ is the Saint-Robert nomographic invariant, the values of the coefficients $\alpha_1, \alpha_2, \beta_2, \gamma_3$ can always be chosen so that $z = f(x, y)$ will be locally nomographed with accuracy up to small quantities of the 5th order by a Cauchy nomogram.

In the present note, by analogy with ([3]), questions are considered concerning the nomographing of a function $z = f(x, y)$ with accuracy up to small quantities of the 6th order by a nomogram with one rectilinear scale, and concerning nomographing with accuracy up to small quantities of the 6th order by a Clark nomogram.

Theorem 1. *A function $z = f(x, y)$ sufficiently smooth at the point x_0, y_0, such that $f'_x(x_0, y_0) \neq 0$, $f'_y(x_0, y_0) \neq 0$, $P(x_0, y_0) \neq 0$, is representable in a neighborhood of this point with accuracy up to small quantities of the 6th order by a nomogram with rectilinear scale x (scale y).*

Proof will be carried out for the case of the scale x. Suppose that, by a suitable admissible transformation ([2]), the function $z = f(x, y)$ is reduced to the form

$$Z = F(X, Y) = X + Y + XY(X - Y)(q_{00} + q_{10}X + \cdots + q_{03}Y^3) + o(\rho^6), \quad (1)$$

where $\rho = \sqrt{X^2 + Y^2}$.

Substituting Z from this formula into the determinant

$$\Delta = \begin{vmatrix} a_1X + a_2X^2 + \cdots & 1 + \alpha_1 X + \alpha_2 X^2 + \cdots & 1 \\ b_1X + b_2Y^2 + \cdots & -1 + \beta_1 Y + \beta_2 Y^2 + \cdots & 1 \\ \frac{1}{2}Z + c_3Z^3 + \cdots & \gamma_2 Z^2 + \gamma_3 Z^3 + \cdots & 1 \end{vmatrix} \tag{2}$$

and equating in the expansion of Δ in powers of X, Y all coefficients of terms of order less than 6 to zero, we obtain the well-known Kreines and Eisenstadt formulas for $a_1, \ldots, \gamma_4$.

Let us now require that the carrier of the scale X be a straight line. Suppose the equation of the carrier of the scale X in the nomogram plane $\xi O \eta$ has the form

$$\eta = k\xi + d. \tag{3}$$

Obviously, then the following relations must hold between the coefficients of the expansions in the first row of the determinant Δ:

$$\alpha_i = ka_i, \qquad i \leq 4. \tag{4}$$

Eliminating k from equations (4), we find

$$\frac{a_1}{a_2} = \frac{\alpha_1}{\alpha_2}, \qquad \frac{a_1}{a_3} = \frac{\alpha_1}{\alpha_3}, \qquad \frac{a_1}{a_4} = \frac{\alpha_1}{\alpha_4}, \qquad \frac{a_2}{a_3} = \frac{\alpha_2}{\alpha_3}. \tag{5}$$

Substituting into (5) the expressions of the required coefficients in terms of $\alpha_1, \alpha_2, \beta_2, \gamma_3$ from Kreĭnes' formulas ([2]), one can write a system of 4 equations with 4 unknowns $\alpha_1, \alpha_2, \beta_2, \gamma_3$:

$$\begin{gathered} \alpha_2 = \frac{1}{2}\alpha_1^2, \\ -\frac{1}{4}\alpha_1^3 - 4\gamma_3 - \frac{1}{2}\alpha_1\beta_2 + \frac{2}{3}q_{00}\alpha_1 + \frac{10}{3}q_{10} - \frac{4}{3}q_{01} = 0, \\ -\frac{1}{8}\alpha_1^4 - 2\alpha_1\gamma_3 - \frac{1}{4}\alpha_1^2\beta_2 + \frac{1}{3}q_{00}\alpha_1^2 + \frac{5}{3}\alpha_1 q_{10} - \frac{2}{3}\alpha_1 q_{01} = 0, \\ -\frac{1}{8}\alpha_1^4 - 2\alpha_1\gamma_3 - \frac{1}{4}\alpha_1^2\beta_2 + \frac{2}{3}q_{00}\alpha_1^2 + \frac{4}{3}q_{00}\beta_2 + \frac{4}{3}\alpha_1 q_{10} \\ + \frac{1}{6}\alpha_1 q_{01} + 4q_{20} + q_{02} - 3q_{11} - 3q_{00}^2 = 0. \end{gathered} \tag{6}$$

If $q_{00} = -\frac{1}{4}P(x_0, y_0)x_0' y_0'$ (3) is different from zero, i.e. $P(x_0, y_0) \neq 0$, then system (6) is consistent and has the unique solution:

$$\alpha_1 = 0, \qquad \alpha_2 = 0, \qquad \beta_2 = \frac{3}{4}\frac{3q_{00}^2 + 3q_{11} - 4q_{20} - q_{02}}{q_{00}}, \qquad \gamma_3 = \frac{5}{6}q_{10} - \frac{1}{3}q_{01}.$$

Since $a_1 = 1$, it follows that $k = 0$, and, moreover, $d = 1$. We now substitute the found values of $\alpha_1, \alpha_2, \beta_2, \gamma_3$ into Kreĭnes' formulas and compute the coefficients $a_1, \ldots, \gamma_4$, necessary for nomographing up to infinitesimals of the 6th order. The coefficient α_5 is found from formula (4).

The proof for the rectilinear scale Y is carried out analogously. It is known (3) that the q_{ik} from (1) are expressed in terms of the nomographic invariants $S(k, m), P, M$. Expressing the necessary coefficients from determinant (2) in terms of these invariants, we can construct this determinant for the function $z = f(x, y)$, bypassing the preliminary reduction of it to the form (1).

For the case of the rectilinear scale X, these coefficients have the form

$$a_1 = 1, \qquad b_1 = 1, \qquad \alpha_1 = 0, \qquad \beta_1 = 0, \qquad a_2 = 0, \qquad b_2 = 0,$$

$$\alpha_2 = 0, \qquad \beta_2 = \frac{1}{4}\frac{S(2,0) + S(1,1) - (P\overline{M})^2}{P\overline{M}}\, x_0'^2,$$

$$\gamma_2 = -\frac{1}{16}\frac{S(2,0) + S(1,1) + 3(P\overline{M})^2}{P\overline{M}}\, x_0'^2,$$

$$a_3 = \frac{1}{24}\frac{S(2,0) + S(1,1) + 5(P\overline{M})^2}{P\overline{M}}\, x_0'^2,$$

$$b_3 = -\frac{1}{24}\frac{5S(2,0) + 5S(1,1) + (P\overline{M})^2}{P\overline{M}}\, x_0'^2,$$

$$c_3 = -\frac{1}{96}\frac{S(2,0) + S(1,1) - (P\overline{M})^2}{P\overline{M}}\, x_0'^2,$$

$$\alpha_3 = 0, \qquad \beta_3 = -\frac{1}{3}[S(1,0) + S(0,1)]x_0'^3, \qquad \gamma_3 = \frac{1}{12}S(1,0)x_0'^3,$$

$$a_4 = \frac{1}{84}[6S(1,0) + S(0,1)]x_0'^3, \qquad b_4 = \frac{1}{84}[22S(1,0) + 15S(0,1)]x_0'^3,$$

$$c_4 = \frac{1}{168}[S(1,0) + S(0,1)]x_0'^3, \qquad \alpha_4 = 0,$$

$$\beta_4 = \left\{ -\frac{1}{24}\left[\frac{S(2,0) + S(1,1) - (P\overline{M})^2}{P\overline{M}}\right]^2 \right.$$

$$\left. + \frac{1}{48}\left[-S(2,0) + 3S(1,1) + 4S(0,2) + 13(P\overline{M})^2\right]\right\} x_0'^4,$$

$$\gamma_4 = \left\{ -\frac{1}{768}\left[\frac{S(2,0) + S(1,1) - (P\overline{M})^2}{P\overline{M}}\right]^2 \right.$$

$$\left. + \frac{1}{192}\left[S(2,0) - 3S(1,1) - 5(P\overline{M})^2\right]\right\} x_0'^4,$$

$$a_5 = \left\{ \frac{1}{480}\left[\frac{S(2,0) + S(1,1) - (P\overline{M})^2}{P\overline{M}}\right]^2 + \right.$$
$$\left. + \frac{1}{720}\left[7S(2,0) + 21S(1,1) - S(0,2) + 41(P\overline{M})^2\right]\right\} x_0'^4,$$

$$b_5 = \left\{ \frac{1}{30}\left[\frac{S(2,0) + S(1,1) - (P\overline{M})^2}{P\overline{M}}\right]^2 + \right.$$
$$\left. + \frac{1}{720}\left[52S(2,0) + 6S(1,1) - 31S(0,2) - 124(P\overline{M})^2\right]\right\} x_0'^4,$$

$$c_5 = \left\{ -\frac{7}{7680}\left[\frac{S(2,0) + S(1,1) - (P\overline{M})^2}{P\overline{M}}\right]^2 - \right.$$
$$\left. - \frac{1}{1440}\left[8S(2,0) + 9S(1,1) + S(0,2) + 4(P\overline{M})^2\right]\right\} x_0'^4.$$

Here $M = -f_y'/f_x'$, $\overline{M} = 1/M$. We find the value $x_0' = 1/f_x'(x_0, y_0)h_1$, where $h_1 \neq 0$ is an arbitrary number.

Theorem 2. If $P(x_0, y_0) = 0$, but $P(x, y) \not\equiv 0$ in any neighborhood of x_0, y_0, then nomographing up to quantities of the 6th order with a rectilinear scale X is possible under the condition

$$S(2,0) + S(1,1) + (P\overline{M})^2 = 0. \tag{7}$$

Proof. In the exceptional case $q_{00} = P(x_0, y_0) = 0$, the system of equations (6) has the form

$$
\begin{gathered}
\alpha_2 = \frac{1}{2}\alpha_1^2, \\
-\frac{1}{4}\alpha_1^3 - 4\gamma_3 - \frac{1}{2}\alpha_1\beta_2 + \frac{10}{3}q_{10} - \frac{4}{3}q_{01} = 0, \\
-\frac{1}{8}\alpha_1^4 - 2\alpha_1\gamma_3 + \frac{1}{4}\alpha_1^2\beta_2 + \frac{5}{3}\alpha_1 q_{10} - \frac{2}{3}\alpha_1 q_{01} = 0, \\
-\frac{1}{8}\alpha_1^4 - 2\alpha_1\gamma_3 - \frac{1}{4}\alpha_1^2\beta_2 + \frac{4}{3}\alpha_1 q_{10} + \frac{1}{6}\alpha_1 q_{01} + 4q_{20} + q_{02} - 3q_{11} = 0.
\end{gathered} \tag{8}
$$

Noting that condition (7) is identically the condition $4q_{20} + q_{02} - 3q_{11} = 0$, we conclude that, when condition (7) is fulfilled, system (8) is consistent and has the solution

$$
\alpha_1 = 0, \qquad \alpha_2 = 0, \qquad \gamma_3 = \frac{5}{6}q_{10} - \frac{1}{3}q_{01}, \qquad \beta_2 \text{ arbitrary.}
$$

The theorem is proved.

This theorem may be proved analogously for the case of a rectilinear scale Y.

Theorem 3. A function $z = f(x, y)$ sufficiently smooth at the point x_0, y_0, such that $f'_x(x_0, y_0) \neq 0$, $f'_y(x_0, y_0) \neq 0$, $P(x_0, y_0) \neq 0$, and for which at this point

$$
[S(2,0) - S(0,2)]\overline{M}P + [S^2(1,0) - S^2(0,1)] + (4\overline{M}P)^3 = 0, \tag{9}
$$

is represented in a neighborhood of x_0, y_0, with accuracy up to quantities of the 6th order, by a Clark nomogram.

Proof. We shall seek the equation of the common carrier for the scales X and Y in the $\xi O\eta$ plane in the form

$$
A_{11}\xi^2 + 2A_{12}\xi\eta + A_{22}\eta^2 + 2A_{13}\xi + 2A_{23}\eta + 1 = 0. \tag{10}
$$

Let ξ and η take the following values: for the scale X, $\xi = a_1X + a_2X^2 + a_3X^3 + a_4X^4$, $\eta = 1 + \alpha_1X + \alpha_2X^2 + \alpha_3X^3 + \alpha_4X^4$; for the scale Y, $\xi = b_1Y + b_2Y^2 + b_3Y^3 + b_4Y^4$, $\eta = -1 + \beta_1Y + \beta_2Y^2 + \beta_3Y^3 + \beta_4Y^4$. Substituting the values of ξ and η for the scale X, and then for the scale Y, into equation (10) and each time equating all coefficients of terms up to the 4th order inclusive to zero, we obtain 10 equations with 9 unknowns $A_{11}, A_{12}, A_{22}, A_{13}, A_{23}, \alpha_1, \alpha_2, \beta_2, \gamma_3$. This system has the form

$$
\begin{gathered}
A_{22}+2A_{23}+1=0, \qquad A_{22}-2A_{23}+1=0, \qquad A_{12}+A_{13}-\alpha_1=0,\\
-A_{12}+A_{13}+\alpha_1=0, \qquad A_{11}+2(\alpha_1^2-\alpha_2)=0, \qquad A_{11}+2(\alpha_1^2+\beta_2)=0,\\
A_{11}\alpha_1+2\alpha_1^3-\alpha_1\alpha_2+\alpha_1\beta_2+8\gamma_3-\frac{4}{3}q_{00}\alpha_1-\frac{20}{3}q_{10}+\frac{8}{3}q_{01}=0,\\
-A_{11}\alpha_1-2\alpha_1^3+\alpha_1\alpha_2-\alpha_1\beta_2-8\gamma_3-\frac{4}{3}q_{00}\alpha_1-\frac{8}{3}q_{10}+\frac{20}{3}q_{01}=0,\\
-\frac{1}{6}\alpha_1^4+\frac{7}{2}\alpha_1^2\alpha_2+\frac{2}{3}\alpha_1^2\beta_2-\frac{11}{3}\alpha_1^2q_{00}-\frac{1}{12}A_{11}\alpha_1^2+4\alpha_1\gamma_3\\
-\frac{7}{3}\alpha_2^2+\frac{1}{3}\alpha_2\beta_2-\frac{8}{3}\alpha_1q_{10}-\frac{1}{3}\alpha_1q_{01}+\frac{2}{3}\alpha_2q_{00}-\frac{8}{3}\beta_2q_{00}\\
+\frac{5}{3}A_{11}\alpha_2+\frac{1}{3}A_{11}\beta_2+6q_{00}^2-2A_{11}q_{00}-8q_{20}-2q_{02}+6q_{11}=0,\\
-\frac{1}{6}\alpha_1^4-\frac{2}{3}\alpha_1^2\alpha_2-\frac{7}{2}\alpha_1^2\beta_2+\frac{11}{3}\alpha_1^2q_{00}-\frac{1}{12}A_{11}\alpha_1^2+4\alpha_1\gamma_3\\
-\frac{7}{3}\beta_2^2+\frac{1}{3}\alpha_2\beta_2-\frac{1}{3}\alpha_1q_{10}-\frac{8}{3}\alpha_1q_{01}-\frac{8}{3}\alpha_2q_{00}+\frac{2}{3}\beta_2q_{00}\\
-\frac{1}{3}A_{11}\alpha_2-\frac{5}{3}A_{11}\beta_2+6q_{00}^2+2A_{11}q_{00}+2q_{20}+8q_{02}-6q_{11}=0.
\end{gathered}
\tag{11}
$$

Taking into account the identity of condition (9) with the condition

$$\frac{7}{2}(q_{10}^2-q_{01}^2)+12q_{00}^2-6q_{00}(q_{20}-q_{02})=0,$$

we observe that, when condition (9) is fulfilled, system (11) is consistent and has the unique solution

$$A_{11}=2(\alpha_2-\alpha_1^2), \qquad A_{12}=\alpha_1, \qquad A_{22}=-1, \qquad A_{13}=0, \qquad A_{23}=0,$$

$$\alpha_1=-\frac{7}{2}\frac{q_{10}-q_{01}}{q_{00}},$$

$$\alpha_2=-\beta_2=\frac{49}{4}\frac{(q_{10}-q_{01})^2}{q_{00}^2}-\frac{15}{2}\frac{q_{20}+q_{02}}{q_{00}}+9\frac{q_{11}}{q_{00}}, \qquad \gamma_3=\frac{1}{4}(q_{10}+q_{01}).$$

The found values $\alpha_1, \alpha_2, \beta_2, \gamma_3$ are substituted into the Kreines formulas, and the coefficients $a_1, \ldots, \gamma_4$, necessary for nomographing up to small quantities of the 6th order, are computed. α_5 and β_5 are found from the condition that in formula (10), for the new ξ and η, the coefficients of the terms of the 5th dimension are zero. The theorem is proved.

Theorem 4. *If* $P(x_0, y_0) = 0$*, but* $P(x, y) \neq 0$ *in any neighborhood of* x_0, y_0*, then nomographing up to small quantities of the 6th order according to Clark is possible under the following conditions: 1) the Bittner invariants are equal to zero; 2)* $S(2,0) + 2S(1,1) + S(0,2) + 2(PM)^2 = 0$*; 3)* $P_x \neq 0$*,* $P_y \neq 0$.

Proof. Consider the system (11′), obtained from (11) for $q_{00} = 0$. Fulfillment of the first condition of the theorem means ([3]) that $q_{10} = q_{01}$. The second condition is identical to the condition $5q_{20} + 5q_{02} - 6q_{11} = 0$. The third condition of the theorem, taking into account the relation between $P(X, Y)$ and $P(x, y)$ and their derivatives, as well as lemma 3 from ([3]), means that $q_{10} = q_{01} \neq 0$. Hence it follows that, when the conditions of the theorem are fulfilled, system (11) is consistent and has the following solutions:

$$A_{11} = 2(\alpha_2 - \alpha_1^2), \qquad A_{12} = \alpha_1, \qquad A_{22} = -1, \qquad A_{13} = 0, \qquad A_{23} = 0,$$

$$\alpha_1 = \frac{6q_{11} - 8q_{20} - 2q_{02}}{q_{10}} = \frac{2q_{20} + 8q_{02} - 6q_{11}}{q_{01}}, \qquad \beta_2 = -\alpha_2,$$

$$\gamma_3 = \frac{1}{2}q_{10} = \frac{1}{2}q_{01}, \qquad \alpha_2 \text{ arbitrary}.$$

The theorem is proved.

The problems solved in the present note, as well as in work ([3]), were posed to the author by S. V. Smirnov and were solved under his scientific guidance. I take this opportunity to express my deep gratitude to S. V. Smirnov.

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Received
29 XI 1957

CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.