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1958

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Abstract

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MATHEMATICS

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ON THE CONVERGENCE OF THE METHOD OF LINES UNDER VARIOUS SCHEMES OF ITS APPLICATION TO THE SOLUTION OF CERTAIN BOUNDARY-VALUE PROBLEMS

(Presented by Academician V. I. Smirnov on 15 VII 1957)

In the literature on the method of lines (^{1}), the possibility of applying it to the solution of boundary-value problems in the plane for nonrectangular domains has been noted more than once; however, in fact the question of convergence of the method has been considered only for a rectangle ($^{2-4}$). It turns out that for nonrectangular domains the known scheme for applying the method of lines (^{1}) is not entirely correct.

The purpose of the present note is to point out the difficulties in applying the former scheme of the method to nonrectangular contours and, in connection with this, to consider a new scheme free of these difficulties.

Let, for example, it be required to solve the Dirichlet problem for the Laplace equation

$$U_{xx} + U_{yy} = 0$$

for a domain bounded by the contour

$$y = Y_1, \quad y = Y_2, \quad Y_1 < Y_2; \qquad x = \varphi_1(y), \quad x = \varphi_2(y), \quad \varphi_1 < \varphi_2, \quad (1)$$

where $\varphi_1(y)$ and $\varphi_2(y)$ are defined and continuous for $Y_1 \le y \le Y_2$.

According to the former scheme, the approximate value of the solution of the Dirichlet problem should be sought on the straight lines $y = y_k = Y_1 + kh$ $(k = 1, 2, \dots, n)$, where

$$h = \frac{Y_2 - Y_1}{n + 1},$$

from the solution of the following boundary-value problem for a system of ordinary differential equations:

Fig. 1

Figure 1: Fig. 1

$$V_k''(x) = \frac{1}{h^2}\left[V_{k-1}(x) - 2V_k(x) + V_{k+1}(x)\right] = 0; \tag{2}$$

$$V_k[\varphi_1(y_k)] = U[\varphi_1(y_k), y_k]; \quad V_k[\varphi_2(y_k)] = U[\varphi_2(y_k), y_k] \quad (k = 1, 2, \dots, n), \tag{3}$$

where $V_0(x) = U(x, Y_1)$, $V_{n+1}(x) = U(x, Y_2)$.

For definiteness, let us consider a contour of the form (1) for which $\varphi_1(y) \equiv 0$, while $\varphi_2(y)$ increases monotonically. Then, solving the system (2) on the interval $[0, \varphi_2(Y_2)]$, we arrive at the necessity, by some unknown means, of continuing the function $U(x, Y_1)$ to the interval $[\varphi_2(Y_1), \varphi_2(Y_2)]$ along the straight line $y = Y_1$. If, on the other hand, one solves the system (2) on the interval $[0, \varphi_2(Y_1)]$, then it is unknown what boundary condition should be imposed on the straight line $x = \varphi_2(Y_1)$ $(Y_1 \leq y \leq Y_2)$.

Let us suppose that, in the general case, the problem of continuing the functions $U(x, Y_1)$ and $U(x, Y_2)$ on the straight lines $y = Y_1$ and $y = Y_2$ has somehow been solved in an appropriate manner. Under this assumption, other difficulties already arise. A detailed consideration of various concrete examples leads to the following results:

i) There exist contours of the form (1) for which the problem (2), (3) for some n may prove unsolvable. Such contours may

may be contours in which the segments $[\varphi_1(Y_1), \varphi_2(Y_1)]$ and $[\varphi_1(Y_2), \varphi_2(Y_2)]$ have no intersection $(n = 2)$; contours symmetric with respect to the axis OY $(n = 4)$, symmetric with respect to both coordinate axes $(n = 9)$; rectangular or isosceles trapezoids $(n = 5)$. (The values of n in parentheses are minimal ones for which problem (2), (3) can already prove to be unsolvable.)

2) For any number N there can always be found such a contour of the form (1) that, for some $n > N$, problem (2), (3) can prove to be unsolvable.

Consider the totality of all contours of the form (1) consisting of the lines $y = 0$, $y = 1$, $x = \varphi_1(y) \equiv 0$, $x = \varphi_2(y)$. Each contour of the totality is completely characterized by the equation of its lateral curve $x = \varphi_2(y) > 0$.

Fig. 1

Therefore, by the word "contour" we shall mean only this lateral curve. Then the set of all contours of this totality may be represented as a part of the metric space $C(0, 1)$, defined by the condition $\varphi_2(y) > 0$ $(0 \leq y \leq 1)$. Denoting it by C, and the set of all those contours from C for which, at least for one n, problem

(2), (3) may have no solution by E, we indicate the following result: the set $E \subset C$ is a set of first category. This result shows that, for almost all contours, problem (2), (3) has a solution for all n. Nevertheless, as yet we know of no specific contour (except a rectangle) for which this circumstance would hold.

In view of the difficulties described, it seems expedient to consider another scheme for applying the method of lines to the solution of boundary-value problems. This new scheme is close in idea to the well-known scheme of I. G. Petrovsky for solving the Dirichlet problem for the Laplace equation by the method of grids (^{5}).

As an example, we shall seek an approximate solution of the Dirichlet problem for the Laplace equation $U_{xx} + U_{yy} = 0$ in the domain G, bounded by the following contour Γ of the form (1): $y = Y_1 = 0,\ y = Y_2,\ x = \varphi_1(y),\ x = \varphi_2(y)$. Construct a contour Γ_δ, also of the form (1), consisting of the lines $y = \delta,\ y = Y_2 - \delta,\ x = \varphi_1(y) + \delta,\ x = \varphi_2(y) - \delta$ for $\delta \le y \le Y_2 - \delta$, where δ is sufficiently small.

We shall denote by $y = y_k$ the straight lines $y = \delta + kh$ $(k = 1, 2, \dots, n)$, where

$$h = \frac{Y_2 - 2\delta}{n+1}.$$

We shall denote by $x_{k,1}$ and $x_{k,2}$ the abscissas of the points of intersection of the straight lines $y = y_k$ with the curves of the contour Γ_δ, and further, where this is necessary, for brevity we shall denote the points themselves in the same way as their abscissas.

We shall consider and solve the k-th equation of system (2) only on the common part $[a_k, b_k]$ of the intervals $[x_{k-1,1}, x_{k-1,2}]$, $[x_{k,1}, x_{k,2}]$, and $[x_{k+1,1}, x_{k+1,2}]$ (having first made h sufficiently small so that the intersection of each three of the mentioned intervals is an interval) (Fig. 1).

We prescribe the boundary conditions for the functions $V_k(x)$ as follows: we put

$$\begin{aligned} V_k(x) &\equiv U[\varphi_1(y_k), y_k] \quad \text{for } x_{k,1} \le x \le a_k; \\ V_k(x) &\equiv U[\varphi_2(y_k), y_k] \quad \text{for } b_k \le x \le x_{k,2} \\ &\qquad (k = 1, 2, \dots, n). \end{aligned} \tag{4}$$

The solution of problem (4) for system (2), with fixed n, is sought by the method of successive approximations, taking as the first approximation $V_k^{(1)}(x)$, for example, functions that are piecewise linear on $[x_{k,1}, x_{k,2}]$ and take, for $a_k \le x \le b_k$, the values

$$V_k^{(1)}(x) = \frac{V_k(a_k) - V_k(b_k)}{a_k - b_k}(x - a_k) + V_k(a_k) \qquad (k = 1, 2, \dots, n);$$

as the second approximation, $V_k^{(2)}(x)$ is the solution of problem (4) for the system

$$V_k''(x) + \frac{1}{h^2}\left[V_{k-1}(x) - 2V_k(x) + V_{k+1}^{(1)}(x)\right] = 0 \qquad (k = 1, 2, \dots, n)$$

(taking into account throughout that $V_0(x) = U(x, 0)$, $V_{n+1}(x) = U(x, Y_2)$), etc. Other variants of the iteration method for solving problem (2), (4) are also possible. In the limit we obtain a system of functions $V_k(x) = \lim\limits_{i\to\infty} V_k^{(i)}(x)$ $(k = 1, 2, \dots, n)$, giving the solution of problem (2), (4).

Denote by $V_k^h(x)$ the value of $V_k(x)$ for a given h. Then, with a corresponding choice $h = h(\delta)$ $\left(\lim\limits_{\delta\to+0} h(\delta) = +0\right)$, the relation

$$\lim_{\delta\to 0} \max_k \max_x |\, U(x, \delta + kh) - V_k^h(x)\,| = 0$$

$$\left(0 \le k \le \frac{Y_2 - 2\delta}{h};\ x_{k,1} \le x \le x_{k,2}\right), \qquad (5)$$

holds, which establishes the convergence of the given scheme of application of the method of lines. In proving equality (5) it is assumed that the exact solution $U(x, y)$ has in G continuous derivatives with respect to y up to the third order inclusive. If one requires from U_{yyy} uniform continuity in G, then the construction can be carried out directly in G, without constructing the inner contour Γ_δ. The scheme described, with small changes, can be applied to the solution of the first boundary-value problem for elliptic equations of the form

$$A(x, y)U_{xx} + B(x, y)U_{yy} + C(x, y)U + D(x, y) = 0$$

under the condition that $AB > 0$, $AC \le 0$ in G, where the given domain is assumed to be bounded by an arbitrary contour (not necessarily by straight lines parallel to the OX axis; the functions $\varphi_1(y)$ and $\varphi_2(y)$ may be discontinuous or multivalued), and also for parabolic equations of the form

$$U_t = A(x, t)U_{xx} + B(x, t)U_x + C(x, t)U + D(x, t)$$

in the case where the domain G is bounded by a contour of the form (1).

A somewhat more modified scheme can be applied to the solution of certain mixed problems for hyperbolic systems of the type

$$U_{i_t} - \lambda_i(x, t)U_{i_x} = \sum_{j=1}^{n} a_{ij}(x, t)U_j + f_i(x, t) \qquad (i = 1, 2, \dots, n)$$

in the domain $0 \leq x \leq l,\ 0 \leq t \leq T$ $(0 < l;\ T > 0)$, in finding both classical and generalized solutions.

In conclusion, I express my gratitude to Prof. A. D. Myshkis, under whose supervision the present work was carried out.

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Received
16 II 1957

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Note: Figure translations are in progress. See original paper for figures.