PHYSICS

V. T. KHOZYAINOV

1958

SovietRxiv

Abstract

Full Text

PHYSICS

V. T. KHOZYAINOV

ON THE THEORY OF MULTIPLE PRODUCTION OF ELEMENTARY EXCITATIONS

(Presented by Academician N. N. Bogolyubov, June 3, 1958)

The state of a quantum liquid (superfluid helium) at temperatures different from absolute zero is a collection of elementary excitations which, in the first approximation, form a gas of quasiparticles (phonons and "rotons"); processes of absorption and emission of excitations (transport and relaxation phenomena) are possible. If, for one reason or another, in some effective volume V of the quantum liquid a certain portion of energy is released or concentrated, elementary excitations must be created. At sufficiently large energies the production will be multiple. This occurs, for example, when a drop of "hot" normal helium enters "cold" superfluid helium, etc. One may assume that in this case a compound system of volume V is formed, which subsequently decays, passing into one of the possible final states characterized by the number of quasiparticles of a definite kind that arise.

Since the specific properties of the interaction leading to the production of excitations are unknown (although at large energy densities the interaction should be large), it is expedient, for calculating the yield of the multiple process, to use Fermi' s statistical method ([1]), based on the assumption of equal excitation of all final states allowed by the conservation laws; the probability of a given yield is then related to the number of states covered by this yield, i.e., to its statistical weight. The relative probability of decay of a compound system of total energy W into n identical quasiparticles is expressed by the formula:

$$S_n(W) = \frac{V^{n-1}}{n!(2\pi\hbar)^{3(n-1)}} \frac{dQ_n(W)}{dW}, \tag{1}$$

where $Q_n(W)$ is the volume of the $3(n-1)$-dimensional momentum space, calculated with allowance for momentum conservation, such that the energy of all quasiparticles is less than W,

$$\frac{dQ_n(W)}{dW} = \frac{d}{dW} \left[\prod_{i=1}^{n} \int dp_i \right] \delta\left(\sum_{i=1}^{n} \mathbf{p}_i \right) U\left(W - \sum_{i=1}^{n} \varepsilon_i \right). \tag{2}$$

Here $U(x) = 1$ for $x > 0$; $U(x) = 0$ for $x < 0$; $\varepsilon(p)$ is the energy of an individual elementary excitation. In obtaining (2) we neglect the law of conservation of angular momentum, which does not introduce an appreciable error in the present approximation. The effective volume of the system V in our case is evidently determined by the conditions of the physical problem. The linear dimensions of the systems of interest to us are bounded from below by the characteristic wavelength of the excitations under consideration (for example, $\lambda = (h/p_0)$ for "rotons"). On the other hand, systems that are too large, whose dimensions substantially exceed the mean free path of an elementary excitation in the surrounding

medium, should behave rather as heterophase inclusions. Owing to the slowness of the processes of diffusion and heat conduction, in this case the process of fragmentation of systems into smaller formations as a result of random actions should play a role.

Using the integral representation of the functions entering into (2), we obtain (see, for example, ([2]))

$$S_n(W) = \frac{V^{n-1}}{n!(2\pi\hbar)^{3(n-1)}} \int_{-\infty-i\delta}^{\infty-i\delta} \frac{d\alpha}{2\pi}\, e^{i\alpha W} \int \frac{d\vec{\lambda}}{(2\pi)^3} \left[\int e^{i\vec{\lambda}\vec{p}-i\alpha\varepsilon(p)}\, d\mathbf{p}\right]^n . \quad (3)$$

This expression can be evaluated exactly in the case of the creation of n phonons (spectrum $\varepsilon = cp$),

$$S_n^{\text{phon}}(W) = \left[\frac{V}{(4\pi)^2(\hbar c)^3}\right]^{n-1} \frac{4(4n-4)!}{(2n-1)!(2n-2)!} \frac{W^{3n-4}}{n!(3n-4)!}, \quad (4)$$

which for $n \gg 1$ can be represented in the form

$$S_n^{\text{phon}}(W) \simeq \frac{2e\sqrt{3}}{(2\pi)^{3/2}W} \frac{n^{3/2}}{n^{4n}} \left[\frac{e^{4\pi}}{(3\pi)^3}\, V \left(\frac{W}{\hbar c}\right)^3\right]^{n-1}, \quad (5)$$

and also in the case of the spectrum $\varepsilon(p) = \Delta + p^2/2\mu$ ("rotons" for $p_0 \to 0$):

$$\Delta S_n^0(w) = \left[\sqrt{\frac{2}{\pi}}\frac{\tau_0}{3}\right]^{n-1} \frac{(w-n)^{3n/2-5/2}}{n^{3/2}n!\,\Gamma\left(\frac{3n}{2}-\frac{3}{2}\right)}, \quad (6)$$

where the notations $w = W/\Delta$, $V = (4\pi/3)R^3$, $\tau_0 = (R/\lambda_0)^3$ have been used, with $\lambda_0 = \hbar/\sqrt{\mu\Delta}$ being the analog of the Compton wavelength of this quasiparticle.

In the general case of "rotons" with the energy spectrum $\varepsilon(p) = \Delta + (p - p_0)^2/2\mu$, the relative probability of creation of n excitations (3) is calculated by successive application of the saddle-point method at each integration, under the assumption that $n \gg 1$. As a result, after rather lengthy calculations we obtain:

$$S_n(w) = \text{const} \cdot [4\tau/3\sqrt{\pi}]^{n-1} F_n(w), \tag{7}$$

where

$$F_n(w) = \left(\frac{e}{2-\theta}\right)^{3n/2} \frac{(\theta e^{\theta})^{n/2}}{n^n} \left(\frac{2-\theta}{n}\right)^{5/2} \frac{[4+8\theta+13\theta^2]^{-3/2}}{\theta\sqrt{2+4\theta-\theta^2}}, \tag{8}$$

where the function $\theta(n)$ is determined by the equation

$$2\theta(n) + 1 + \frac{w-n}{\gamma n} = \sqrt{\left(\frac{w-n}{\gamma n}\right)^2 + 10\left(\frac{w-n}{\gamma n}\right) + 1} \tag{9}$$

and the following additional notations have been introduced: $\gamma = p_0^2/4\mu\Delta$, $\tau = (R/\lambda)^3$, $\lambda = \hbar/p_0$ —the characteristic wavelength of the roton. The first omitted term is, relative to (8), of order $(\sqrt{n})^{-1}$.

In an analogous way one obtains formulas which, in the same approximation, give the probability of simultaneous formation of, for example, s "rotons" and $n-s$ phonons. In this case probability (3) contains under the integral the product of the factors $\int \exp[i\lambda p - i\alpha\varepsilon(p)]\, dp$ in the corresponding powers. We give the formula obtained under the assumption $w - s \gg 2\gamma$, which corresponds to the case when the number of rotons is substantially smaller than the number of phonons:

$$S_{s,n-s}(w) = \frac{\text{const}}{n!} \left[\frac{V}{\pi^2}\left(\frac{\Delta}{\hbar c}\right)^3\right]^{n-s} \left[\sqrt{2}\left(\frac{\mu\Delta}{\pi\hbar^2}\right)^{3/2} V\right]^{s-1} \frac{(w-s)^{3n-3s/2-5/2}}{\Gamma\left(3n - \frac{3s}{2} - \frac{3}{2}\right)}. \tag{10}$$

Using the probabilities found above, one can obtain the mean number of elementary excitations of a given kind formed in the decay of a composite system

$$\bar{n} = \sum nS_n \Big/ \sum S_n, \tag{11}$$

as well as the quadratic deviation from the mean; here the summation in (11) is over all possible values of n, i.e. from $n = 2$ (the minimum number of quasiparticles required for satisfaction of the momentum-conservation law) to $n_{\max}(w)$,

Fig. 1

Figure 1: Fig. 1

determined by the available energy. Since all formulas for the relative probabilities have the form const $\cdot a^{n-1} F_n(w)$, one may write

$$\bar{n} = a \frac{d}{da} \ln Z,$$

$$\overline{(n-\bar{n})^2} = a \frac{d}{da} \left(a \frac{d}{da} \ln Z \right), \tag{12}$$

where

$$Z(a, w) = \sum a^n F_n(w) = a \sum S_n. \tag{13}$$

Fig. 1

The sum Z is not evaluated exactly. However, making use of the fact that the relative probability S_n as a function of n has a sharp maximum within the interval of variation of n, we can apply the Euler–Maclaurin formula with sufficient accuracy.

If the function $\varphi(x)$ is defined so that $S(x) = \text{const} \cdot a^{xF}(x) = \text{const} \cdot \exp \varphi(x)$, then the point x_0 of the maximum of the function $S(x)$ is found from the equation

$$\varphi'(x_0) = \ln a + (\ln F(x_0))' = 0, \tag{14}$$

and $\varphi''(x_0) < 0$ no longer depends on the parameter a. Expanding $\varphi(x)$ in a series near the point of maximum x_0, we find

$$\int_2^{x_{\max}} e^{\varphi(x)}\, dx = \frac{\sqrt{2\pi}}{\sqrt{-\varphi''(x_0)}}\, e^{\varphi(x_0)} \left\{ 1 + O\left(\frac{1}{x_0}\right) \right\}, \tag{15}$$

whence we obtain

$$\bar{n} = x_0 \left\{ 1 + O\left(\frac{1}{x_0}\right) \right\}, \qquad \overline{(n-\bar{n})^2} = [-\varphi''(x_0)]^{-1} \left\{ 1 + O\left(\frac{1}{x_0}\right) \right\}. \tag{16}$$

The matter is thus reduced to solving the transcendental equation (14), whose form depends on the energy spectrum of the generated excitations.

Fig. 2

Figure 2: Fig. 2

For phonons we have the equation:

$$\ln \frac{e^4}{(3\pi)^3} V \left(\frac{W}{\hbar c}\right)^3 - 4 \ln x_0 + \frac{3}{2x_0} - 4 = 0, \tag{17}$$

which leads to the expression

$$\frac{\overline{n(E)}}{V} = \frac{1}{\sqrt{\pi}} \left(\frac{E}{3\hbar c}\right)^{3/4} \left\{1 + \frac{3\sqrt{\pi}}{8V} \left(\frac{3\hbar c}{E}\right)^{3/4} + ...\right\}, \tag{18}$$

where $E = W/V$ is the energy density in the composite system.

In the case of the formation of "rotons," the use of formulas (7), (8), (9) leads to the relations

$$\frac{\overline{n(w)}}{\tau} = \frac{w}{\tau} \zeta \left(\frac{w}{\tau}\right), \qquad \ln \frac{w}{\tau} = \Phi(\zeta), \tag{19}$$

where the function $\Phi(\zeta)$ is defined by the expression

$$\Phi(\zeta) = \ln \frac{4\sqrt{\theta}}{3\sqrt{\pi}\,\zeta(2-\theta)^{3/2}} + \frac{\theta+1}{2} + \frac{2+4\theta-\theta^2}{2\theta\sqrt{(1-\zeta)^2 + 10\gamma\zeta(1-\zeta) + \gamma^2\zeta^2}}, \tag{20}$$

where

$$2\gamma\zeta\theta(\zeta) = \sqrt{(1-\zeta)^2 + 10\gamma\zeta(1-\zeta) + \gamma^2\zeta^2} - (\gamma-1)\zeta - 1. \tag{21}$$

The quantity $\overline{n(w)}/\tau$ is the mean number of excitations formed per part of the "active" volume of magnitude $(4\pi/3)\lambda^3$. The course of the curve

Fig. 2

$\overline{n(w)}/\tau$ on a logarithmic scale is shown in Fig. 1. In the limiting case of large energy densities in the system ($\gamma\zeta \ll 1$), we obtain the asymptotic formula

$$\frac{\overline{n(w)}}{\tau_0} = 0.805 \left(\frac{w}{\tau_0}\right)^{3/5} \left\{1 + 9.15 \left(\frac{\tau_0}{w}\right)^{2/5} + \cdots\right\}. \tag{22}$$

An analogous limiting law is obtained in calculations with $p_0 \to 0$; consequently, at large energies the characteristic role is played by the wavelength λ_0, and the difference between the spectra is erased. The course, calculated for the general case, of the derivative $d\ln(\bar{n}/\tau)/d\ln(w/\tau)$ is shown in Fig. 2, where the transition to the limiting power law is clearly visible.

Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

Received
23 V 1958

CITED LITERATURE

1. E. Fermi, Prog. Theor. Phys., **5**, 750 (1950).
2. J. V. Lepore, R. N. Stuart, Phys. Rev., **94**, 1724 (1954).

Note: Figure translations are in progress. See original paper for figures.