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Abstract

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MATHEMATICS

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ON THE APPLICATION OF CERTAIN INTEGRO-DIFFERENTIAL OPERATORS

In the present note we give formulations of a number of new results in the theory of Dirichlet series and in the theory of quasianalytic classes of functions. These results are obtained by introducing special integro-differential operators connected with the concept of fractional integration in the sense of Riemann–Liouville or H. Weyl.

1°. Let the function $F(\sigma)$ be defined and continuous on the half-axis $(\sigma_0, +\infty)$. For any $\alpha > 0$ define the operator

$$\frac{d_e^{-\alpha} F(\sigma)}{d_e \sigma^{-\alpha}} \equiv \frac{1}{\Gamma(\alpha)} \int_\sigma^{+\infty} (e^{-\sigma} - e^{-u})^{\alpha-1} e^{-u} F(u)\, du, \tag{1}$$

calling it the fractional integral of the function $F(\sigma)$ of order α with endpoint at $+\infty$. It is easy to see that

$$\lim_{\alpha \to +0} \frac{d_e^{-\alpha} F(\sigma)}{d_e \sigma^{-\alpha}} \equiv F(\sigma),$$

and therefore it is natural to regard the function $F(\sigma)$ itself as the integral of order zero.

Let the sequence $\{\mu_n\}$ $(n \geqslant 0)$ satisfy the condition

$$\mu_0 = 0; \qquad 0 < \mu_{k+1} - \mu_k \leqslant 1 \quad (k \geqslant 0); \qquad \lim_{k\to\infty} \mu_k = +\infty. \tag{2}$$

Denote $\alpha_k = 1 - (\mu_k - \mu_{k-1})$ $(k = 1, 2, \ldots)$, and introduce the operators

$$L^{(\mu_0)} F(\sigma) \equiv F(\sigma), \qquad L^{(\mu_k)} F(\sigma) \equiv -\frac{d_e^{-\alpha_k}}{d_e \sigma^{-\alpha_k}} \left\{ e^\sigma \frac{d}{d\sigma} L^{(\mu_{k-1})} F(\sigma) \right\} \quad (k \geqslant 1), \tag{3}$$

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assuming that all of them exist and are continuous on the half-axis $(\sigma_0, +\infty)$.

We shall agree to say that the function $L^{(\mu_k)}F(\sigma)$ is continuous on $(\sigma_0, +\infty]$ if: 1) it is continuous on the interval $(\sigma_0, +\infty)$; 2) there exists a finite limit

$$L^{(\mu_k)}F(+\infty) = \lim_{\sigma\to+\infty} L^{(\mu_k)}F(\sigma).$$

We shall say that $F(\sigma) \in \mathcal{L}(\mu_n; \sigma_0)$ if all the functions $L^{(\mu_k)}F(\sigma)$ $(k \geqslant 0)$ exist and are continuous on $(\sigma_0, +\infty]$, while the functions

$$e^{\sigma}\frac{d}{d\sigma}L^{(\mu_k)}F(\sigma) \quad (k = 0, 1, 2, ...)$$

are continuous and absolutely integrable on $(\sigma_1, +\infty)$, where $\sigma_1 > \sigma_0$ is arbitrary.

Theorem 1. If $F(\sigma) \in \mathcal{L}(\mu_n; \sigma_0)$, then for any $n \geqslant 0$ and $\sigma \in (\sigma_0, +\infty]$ the formula holds

$$F(\sigma) = \sum_{k=0}^{n} \frac{L^{(\mu_k)}F(+\infty)}{\Gamma(1+\mu_k)} e^{-\mu_k\sigma} + \frac{1}{\Gamma(\mu_{n+1})}\int_{\sigma}^{+\infty} (e^{-\sigma} - e^{-u})^{\mu_{n+1}-1} e^{-u} L^{(\mu_{n+1})}F(u)\, du. \tag{4}$$

This formula is, in a certain sense, an analogue of Taylor’ s formula and, as is not difficult to see, in the particular case when $\mu_n = n$ $(n = 0, 1, 2, ...)$, after the change of variable $e^{-\sigma} = x$, coincides with it.

From the class $\mathcal{L}(\mu_n; \sigma_0)$ we single out the subclass $\mathcal{L}^*(\mu_n; \sigma_0)$ of those functions $F(\sigma)$ for which, as $n \to \infty$, the integral remainder term in formula (4) tends uniformly to zero in any interval $[\sigma_1, +\infty] \subset (\sigma_0, +\infty]$. Consequently, if $F(\sigma) \in \mathcal{L}^*(\mu_n; \sigma_0)$, then the expansion into a Dirichlet series is valid:

$$F(\sigma) = \sum_{k=0}^{\infty} \frac{L^{(\mu_k)}F(+\infty)}{\Gamma(1+\mu_k)} e^{-\mu_k\sigma}, \tag{5}$$

uniformly convergent on any half-line $[\sigma_1, +\infty] \subset (\sigma_0, +\infty]$. It is clear, moreover, that if $F(\sigma) \in \mathcal{L}^*(\mu_n; \sigma_0)$, then it admits an analytic continuation to the whole half-plane $\sigma = \operatorname{Re} s > \sigma_0$ $(s = \sigma + it)$, and the expansion (5) remains valid in the same half-plane.

The following criterion of necessary-and-sufficient type holds for the expansibility of functions in a Dirichlet series with respect to the given system $\{e^{-\mu_k s}\}$.

Theorem 2. *Let the sequence* $\{\mu_n\}$ $(n \geq 0)$ *satisfy condition (1).*

a) *If* $F(\sigma) \in \mathcal{L}(\mu_n; \sigma_0)$ *and, in addition,*

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$$\sup_{(\sigma_0,+\infty]} \left|L^{(\mu_k)}F(\sigma)\right| \leq Me^{-\sigma_0\mu_k}\Gamma(1+\mu_k) \quad (k \geq 0),$$

then $F(\sigma) \in \mathcal{L}^*(\mu_n;\sigma_0)$*, i.e. the expansion*

$$F(\sigma) = \sum_{k=0}^{\infty} \frac{L^{(\mu_k)}F(+\infty)}{\Gamma(1+\mu_k)} e^{-\mu_k\sigma}, \qquad \sigma \in (\sigma_0, +\infty],$$

holds.

b) *If, in addition to (1), the sequence* $\{\mu_n\}$ *also satisfies the condition*

$$\limsup_{k\to\infty} \frac{\log k}{\mu_k} < l < +\infty \tag{6}$$

and the expansion

$$F(\sigma) = \sum_{k=0}^{\infty} a_k e^{-\mu_k\sigma}, \qquad \sigma \in (\sigma_0, +\infty],$$

holds, then for $\sigma_1 = \sigma_0 + l$ *one has* $F(\sigma) \in \mathcal{L}(\mu_n;\sigma_1)$*; moreover,*

$$\sup_{(\sigma_1,+\infty]} \left|L^{(\mu_k)}F(\sigma)\right| \leq AB^{\mu_k}\Gamma(1+\mu_k) \quad (k \geq 0),$$

$$a_k = \frac{L^{(\mu_k)}F(+\infty)}{\Gamma(1+\mu_k)} \quad (k \geq 0).$$

In the case when the condition $0 < \mu_{k+1} - \mu_k \leq 1$ $(k \geq 0)$ is not fulfilled, the following result holds.

Theorem 3. *Let the sequence* $\{\mu_n\}$ $(n \geq 0)$ *satisfy the conditions*

$$\mu_0 \geq 0; \qquad 0 < \mu_{k+1} - \mu_k \quad (k \geq 0); \qquad \lim_{k\to\infty} \mu_k = +\infty. \tag{7}$$

a) *If* $F(\sigma)$ *is defined on* $(\sigma_0, +\infty)$ *and there exists an extension of* $\{\mu_n\}$ *to a sequence* $\{\mu_n^*\}$ *satisfying conditions (1), such that:*

1) $F(\sigma) \in \mathcal{L}(\mu_n^*;\sigma_0)$;

2)

$$\sup_{(\sigma_0,+\infty]} \left|L^{(\mu_k)}F(\sigma)\right| \leq Me^{-\sigma_0\mu_k}\Gamma(1+\mu_k) \quad (k \geq 0);$$

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3) $L^{(\mu_k^*)}F(+\infty) = 0$, if $\mu_k^* \notin \{\mu_n\}$,

then the expansion into a Dirichlet series is valid:

$$F(\sigma) = \sum_{k=0}^{\infty} \frac{L^{(\mu_k)}F(+\infty)}{\Gamma(1+\mu_k)} e^{-\mu_k \sigma}, \qquad \sigma \in (\sigma_0, +\infty].$$

In this case $F(\sigma)$ can be analytically continued to the half-plane $\sigma = \operatorname{Re} s > \sigma_0$ ($s = \sigma + it$), and the expansion remains valid in the whole half-plane $\sigma = \operatorname{Re} s > \sigma_0$.

b) Let $\{\mu_n\}$ ($n \geq 0$) satisfy conditions (7) and (6), and suppose that there is an expansion

$$F(\sigma) = \sum_{k=0}^{\infty} a_k e^{-\mu_k \sigma}, \qquad \sigma \in (\sigma_0, +\infty).$$

For any completion of $\{\mu_n\}$ to a sequence $\{\mu_n^*\}$ satisfying conditions (1) and

$$\limsup_{k \to +\infty} \frac{\log k}{\mu_k^*} < l^* < +\infty,$$

the following assertions hold for $\sigma^* = \sigma_0 + l^*$:

1) $F(\sigma) \in \mathcal{L}(\mu_n^*; \sigma^*)$;

2)

$$\sup_{(\sigma^*, +\infty)} \left| L^{(\mu_k)}(F(\sigma)) \right| \leq AB^{\mu_k} \Gamma(1+\mu_k) \qquad (k \geq 0);$$

3)

$$a_k = \frac{L^{(\mu_k)}F(+\infty)}{\Gamma(1+\mu_k)} \qquad (k \geq 0),$$

where $L^{(\mu_k)}F(\sigma) = L^{(\mu_{n_k}{}^*)}F(\sigma)$, if $\mu_k = \mu_{n_k}^*$.

2°. For a function $f(x)$, defined and continuous on the half-line $[0, +\infty)$, and for a given sequence $\{\alpha_k\}$ ($k \geq 0$), where $0 \leq \alpha_k < 1$ ($k \geq 0$), we introduce the operators

$$D^0 f(x) \equiv \frac{d^{-\alpha_0}}{dx^{-\alpha_0}} f(x); \qquad D^{(k)} f(x) \equiv \frac{d^{-\alpha_k}}{dx^{-\alpha_k}} \frac{d}{dx} D^{(k-1)} f(x) \qquad (k \geq 0),$$

where, for $\alpha > 0$,

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$$\frac{d^{-\alpha}}{dx^{-\alpha}}f(x) \equiv \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)\,dt$$

is the fractional integral of order α of the function $f(x)$ in the Riemann-Liouville sense.*

We shall call the function $D^{(k)}f(x)$ (if it exists) the k-th generalized derivative of $f(x)$ with respect to the given sequence $\{\alpha_k\}$. We shall say that $f(x) \in C\{\alpha_k\}$ if the functions $D^{(k)}f(x)$ $(k \geq 0)$ are continuous on the half-line $[0, +\infty)$, and the functions

$$\frac{d}{dx}D^{(k)}f(x) \qquad (k \geq 0)$$

are continuous on $(0, +\infty)$ and absolutely integrable on every interval $[0, \delta]$ $(\delta \geq 0)$.

Let $\{m_n\}$ $(n \geq 0)$ be some sequence of positive numbers. We assign to the class $C_{m_n}\{\alpha_k\}$ all those functions $f(x)$ from the class $C\{\alpha_k\}$ for which:

a)

$$\left|D^{(k)}f(x)\right| \leq AB^{\sum_1^k(1-\alpha_i)}\,m_k e^{Cx} \qquad (k \geq 0), \quad 0 \leq x < +\infty,$$

where A, B, and C are constants depending on the given function $f(x)$;

b)

$$e^{-Cx}\left|\frac{d}{dx}D^{(k)}f(x)\right| \in L_1(0, +\infty) \qquad (k \geq 0).$$

We shall say that the set of functions $C_{m_n}\{\alpha_k\}$ constitutes a quasi-analytic class in the generalized sense if, for any functions $f_1(x)$ and $f_2(x) \in C_{m_n}\{\alpha_k\}$, the equalities $D^{(k)}f_1(0) = D^{(k)}f_2(0)$ $(k \geq 0)$ imply the identity

$$f_1(x) \equiv f_2(x), \qquad 0 \leq x < +\infty.$$

The following assertion holds—an analogue of the well-known Carleman-Ostrovsky theorem.

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* It is easy to see the connection between the operators $L^{(\mu_k)}F(\sigma)$ and $D^{(k)}f(x)$, if one makes the change of variable $x = e^{-\sigma}$.

Theorem 4. For the quasianalyticity of the class $C_{m_n}\{\alpha_k\}$ it is necessary and sufficient that

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$$\int_1^{+\infty} \frac{\log T_\alpha(r)}{r^2}\,dr = +\infty,$$

where

$$T_\alpha(r) = \sup_{n\geq 1} \frac{r^{\sum_1^n (1-\alpha_k)}}{m_n}.$$

3°. Let $f(x)$ be given on $(-\infty, +\infty)$.

The **Weyl integral of order** $\alpha > 0$ of the function $f(x)$ is the function

$$W^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} (x-t)^{\alpha-1} f(t)\,dt.$$

It is natural to set $W^0 f(x) \equiv f(x)$.

For a given sequence $\{\alpha_k\}$ $(k \geq 0;\ 0 \leq \alpha_k < 1)$ we introduce the operations

$$R^{(0)} f(x) \equiv W^{\alpha_0} f(x); \qquad R^{(k)} f(x) \equiv W^{\alpha_k} \frac{d}{dx} R^{(k-1)} f(x) \quad (k \geq 1).$$

It is easy to see that if $\alpha_k = 0$ $(k \geq 0)$, then $R^{(k)} f(x) \equiv f^{(k)}(x)$.

Let the function $p(x)$ be defined and continuously differentiable on the half-axis $[0, +\infty)$, and let $\lim_{t\to+\infty} p'(t) = +\infty$, while the function $q(x)$, as in ([1]), is conjugate to $p(x)$ in the sense of Young. Let, further, $\{m_n\}$ be some sequence of positive numbers.

We assign to the class $C_{m_n}\{p(x); \alpha_k\}$ all functions satisfying the conditions:

a) The functions $R^{(k)} f(x)$ and $\frac{d}{dx} R^{(k)} f(x)$ exist and are continuous on the whole axis $(-\infty, +\infty)$;

b)

$$|R^{(k)} f(x)| \leq m_k \omega_f(x) e^{-p_1(x)} \quad (k = 0, 1, 2, \dots), \qquad -\infty < x < +\infty,$$

where $\omega_f(x) \geq 0$ is summable on $(-\infty, +\infty)$, and $p_1(x) \equiv p(|x|)$ for $x \leq 0$, while $p_1(x) \geq C_0$ for $x > 0$ (C_0 is a real constant);

c)

$$\left|(1+x^2)\frac{d}{dx}R^{(k)}f(x)\right| \le C_k \quad (k=0,1,2,...), \qquad -\infty < x < +\infty,$$

where $C_k > 0$ are some constants.

Theorem 5. The class $C_{m_n}\{p(x);\alpha_k\}$ is empty, in other words, contains only the function $f(x) \equiv 0$, $-\infty < x < +\infty$, if

$$\lim_{R\to+\infty} \inf \left\{ \frac{q(R)}{R} - \frac{2}{\pi}\int_1^R \frac{\log T_\alpha(r)}{r^2}\, dr \right\} = -\infty,$$

where

$$T_\alpha(r) = \sup_{n\ge 1} \frac{r^{\sum_1^n(1-\alpha_k)}}{m_n}.$$

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1. M. M. Dzhrbashyan, *Izv. AN ArmSSR, ser. phys.-mat. sciences*, **10**, No. 6, 7 (1957).

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