APPLICATION OF THE THEORY OF ENTIRE FUNCTIONS TO THE CONSTRUCTION AND STUDY OF \(N’\)-FUNCTIONS COMPLEMENTARY TO GIVEN \(N’\)-FUNCTIONS

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Abstract

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MATHEMATICS

V. S. VIDENSKII

APPLICATION OF THE THEORY OF ENTIRE FUNCTIONS TO THE CONSTRUCTION AND STUDY OF N'-FUNCTIONS COMPLEMENTARY TO GIVEN N'-FUNCTIONS

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1. In the theory of Orlicz spaces ([1]), an important role is played by the class of even convex functions defined on the entire real axis, called N'-functions and possessing the property that every N'-function $M(u)$ can be written in the form

$$M(u) = \int_0^{|u|} p(t)\,dt, \tag{1}$$

where $p(t)$ is a right-continuous nondecreasing function such that

$$p(0) = 0, \qquad p(t) > 0 \quad \text{for } t > 0, \qquad \lim_{t\to\infty} p(t) = \infty. \tag{2}$$

If we put

$$q(s) = \sup_{p(t)\le s} t, \tag{3}$$

then the N'-function

$$N(v) = \int_0^{|v|} q(s)\,ds \tag{4}$$

will be called the function complementary to $M(u)$. But, since it follows from (3) that

$$p(t) = \sup_{q(s)\le t} s, \tag{5}$$

$M(u)$, in turn, is complementary to $N(v)$. The functions $M(u)$ and $N(v)$ are, moreover, connected by the relations

$$N(v) = \max_{u \geq 0} [u|v| - M(u)], \tag{6}$$

$$M(u) = \max_{v \geq 0} [v|u| - N(v)]. \tag{7}$$

We note that from (1) and (4) it follows that

$$\lim_{u\to\infty} u^{-1}M(u) = \infty, \qquad \lim_{v\to\infty} v^{-1}N(v) = \infty; \tag{8}$$

it is known, moreover, that this tendency is monotone.

Two N'-functions $M_1(u)$ and $M_2(u)$ are called equivalent if there exist constants α, β and u_0 such that

$$M_1(\alpha u) \leq M_2(u) \leq M_1(\beta u) \qquad (u \geq u_0). \tag{9}$$

In particular, the N'-functions $M_1(u)$ and $M_2(u)$ are equivalent if

$$\lim_{u\to\infty} \frac{M_1(u)}{M_2(u)} = k \qquad (0 < k < \infty). \tag{10}$$

Starting from (4) or (6), the determination of an explicit expression for the complementary function $N(v)$ for a given N'-function $M(u)$ is in many cases difficult; therefore, instead of $N(v)$, one tries to find an equivalent function $N_1(v)$. Under various assumptions concerning the rate of growth of $M(u)$, this problem was studied by M. A. Krasnosel' skii and Ya. B. Rutickii ([1,2]). We shall consider this question in its general form, imposing no restrictions on the N'-function $M(u)$.

In all that follows, by $F(z)$ we shall denote the entire function*

$$F(z) = \sum_{n=0}^{\infty} e^{-M(n)} z^n. \tag{11}$$

Theorem 1. *If the N'-function $N(v)$ is complementary to the N'-function $M(u)$, then the function $\ln F(e^v)$ is equivalent to $N(v)$.**

Let $\mu(r)$ denote the maximum term of the function $F(z)$; evidently, the equality

$$\ln \mu(r) = \max_n [\, n \ln r - M(n)\,] \qquad (r \geqslant 1), \tag{12}$$

holds, where the maximum is taken over all natural n. If we put $\ln r = v$ and $\ln \mu(r) = N_0(v)$, then from (6) and (12) it follows that $N_0(v) \leqslant N(v)$. Next, suppose that for some fixed v the maximum in (6) is attained at $u = u'$, where u' is not a natural number; consequently $n' < u' < n' + 1$, where $n' = [u']$. It is clear that

$$N_0(v) \leqslant N(v) = u'v - M(u') < (n'+1)v - M(n') \leqslant N_0(v) + v, \tag{13}$$

whence, taking (8) into account, we shall have

$$\lim_{v\to\infty} \frac{N(v)}{N_0(v)} = 1. \tag{14}$$

Since (see, for example, ([4]), Section IV, problem 33)

$$\ln \mu(r) = \int_0^r t^{-1}\nu(t)\,dt, \tag{15}$$

where $\nu(t)$ is a positive function, continuous from the right and tending monotonically to infinity as $t \to \infty$, it follows that $N_0(v)$ will be an N'-function. It remains to verify that $N_0(v)$ is equivalent*** to $\ln F(e^v)$. Indeed, from (12) it follows that, for arbitrary λ $(0 < \lambda < 1)$ and $v \geqslant v_0(\lambda)$, the inequalities

$$e^{N_0(v+\ln\lambda)} < F(\lambda e^v) = \sum_{n=0}^{\infty} e^{-M(n)} e^{nv} \lambda^n < \frac{e^{N_0(v)}}{1-\lambda}, \tag{16}$$

hold, whence, for $v \geqslant v_1(\lambda)$, it follows that

$$N_0(\lambda v) < \ln F(e^v) < N_0\left(\frac{v}{\lambda}\right). \tag{17}$$

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* The function $F(z)$ will be entire by virtue of (8).

** The fact that $\ln F(e^v)$ for $v > 0$ is an N'-function is easily checked ([3]).

*** Incidentally, this fact can be derived from inequalities established by me earlier (([5]), p.218).

From (14) and (17) we obtain Theorem 1.

2. If one assumes that the entire function $F(z)$, defined by formula (11), is of finite order, then regarding the connection between the mutually complementary N'-functions $M(u)$ and $N(v)$ one can state assertions of a different kind than those obtained earlier in the study of Orlicz spaces.

Theorem 2. *If $F(z)$ is an entire function of finite order, then there exists the limit*

$$\lim_{v\to\infty} \frac{\ln F(e^v)}{N(v)} = 1. \qquad (18)$$

Relation (18) follows from (14) and the known equality (see (4))

$$\lim_{r\to\infty} \frac{\ln F(r)}{\ln \mu(r)} = 1, \qquad (19)$$

which is valid for entire functions of finite order.

Theorem 3. *In order that*

$$\overline{\lim}_{v\to\infty} v^{-1} \ln N(v) = \rho \qquad (0 \le \rho \le \infty), \qquad (20)$$

it is necessary and sufficient that

$$\overline{\lim}_{n\to\infty} \frac{M(n)}{n \ln n} = \frac{1}{\rho}, \qquad (21)$$

where n runs through the natural numbers.

In order that

$$\overline{\lim}_{v\to\infty} e^{-\rho v} N(v) = \sigma \qquad (0 < \rho < \infty,\ 0 < \sigma < \infty), \qquad (22)$$

it is necessary and sufficient that

$$\overline{\lim}_{n\to\infty} n^{1/\rho} e^{-M(n)/n} = (\sigma e \rho)^{1/\rho}, \qquad (23)$$

where n runs through the natural numbers.

The theorem follows from the known connection between the growth of an entire function and the rate of decrease of the coefficients of its power series ((6), p. 500).

If, instead of (22), one assumes the existence of the limit

$$\lim_{v\to\infty} e^{-\rho v} N(v) = \sigma \qquad (0 < \rho < \infty,\ 0 < \sigma < \infty), \qquad (24)$$

then from (7) it will follow that there exists the limit

$$\lim_{u\to\infty}\frac{M(u)}{u\ln u}=\frac{1}{\rho}. \tag{25}$$

Conversely, (25) entails the existence of a finite limit for $e^{-\rho v}N(v)$. But the following somewhat more precise assertion is also true.

Theorem 4. *In order that (24) hold, it is necessary and sufficient that there exist the limit*

$$\lim_{u\to\infty} u^{1/\rho}e^{-M(u)/u}=(\sigma e\rho)^{1/\rho}. \tag{26}$$

Lemma. *Let*

$$f(z)=\sum_{n=0}^{\infty}a_n z^n. \tag{27}$$

an entire function with positive coefficients, which satisfy the inequalities

$$\frac{a_1}{a_0}\geq\frac{a_2}{a_1}\geq\cdots\geq\frac{a_{n+1}}{a_n}\geq\cdots. \tag{28}$$

In order that the condition

$$\lim_{r\to\infty} r^{-\rho}\ln f(r)=\sigma \quad (0<\rho<\infty,\ 0<\sigma<\infty), \tag{29}$$

be fulfilled, it is necessary and sufficient that

$$\lim_{n\to\infty} n^{1/\rho}a_n^{1/n}=(\sigma e\rho)^{1/\rho}. \tag{30}$$

The necessity of condition (30) was proved in ([4]) (Section IV, Problem 69). Let us prove the sufficiency of condition (30). Indeed, for any $\varepsilon>0$,

$$\left(\frac{\sigma' e\rho}{n}\right)^{n/\rho}<a_n<\left(\frac{\sigma'' e\rho}{n}\right)^{n/\rho}, \quad n>N(\varepsilon). \tag{31}$$

From the right-hand side of inequality (31) it follows that

$$f(r)<e^{(\sigma''+\varepsilon)r^\rho}, \quad r>R(\varepsilon). \tag{32}$$

On the other hand, it is clear that

$$\left(\frac{\sigma' e\rho}{n}\right)^{n/\rho} r^n < f(r) = \sum_{n=0}^{\infty} a_n r^n, \quad n > N(\varepsilon). \tag{33}$$

Putting $n = \rho\sigma' r^\rho$ in the left-hand side of (33), we find

$$e^{\sigma' r^\rho} < f(r), \quad r > R(\varepsilon). \tag{34}$$

Comparing (32) and (34), we obtain the limiting relation (29). In view of the convexity of the function $M(u)$, condition (28) is fulfilled for the coefficients of the function $F(z)$.

Theorem 5. The integrals

$$\int_1^\infty e^{-\rho v} N(v)\, dv, \quad \int_1^\infty e^{-\rho M(u)} u\, du \quad (0 < \rho < \infty) \tag{35}$$

converge and diverge simultaneously.

By Valiron's theorem ([7]), thanks to (28), there is equiconvergence of the integral and the series

$$\int_1^\infty r^{-\rho-1} \ln F(r)\, dr, \quad \sum_{n=1}^{\infty} e^{-\rho[M(n+1)-M(n)]}. \tag{36}$$

But this series converges and diverges simultaneously with the series

$$\sum_{n=1}^{\infty} e^{-\rho M(n)/n}. \tag{37}$$

Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

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Note: Figure translations are in progress. See original paper for figures.