Mathematics

P. E. SOBOLEVSKII

1958

SovietRxiv

Abstract

Full Text

Mathematics

P. E. SOBOLEVSKII

ON FIRST-ORDER DIFFERENTIAL EQUATIONS IN A HILBERT SPACE WITH A VARIABLE POSITIVE-DEFINITE SELF-ADJOINT OPERATOR WHOSE FRACTIONAL POWER HAS A CONSTANT DOMAIN OF DEFINITION

(Presented by Academician I. G. Petrovskii, August 1, 1958)

1. In recent years a number of authors have studied the problem

$$v' + A(t)v = 0 \qquad (0 \le t \le 1), \tag{1}$$

$$v(0) = v_0 \tag{2}$$

in a Hilbert space, to which various boundary-value problems for equations of parabolic type reduce ([1−12]).

In ([1,2,5−12]) it was assumed that the domain of definition $D[A(t)]$ of the operator $A(t)$ does not depend on t. The case of equation (1) with an operator $A(t)$ having a variable domain of definition, considered in ([13]), is reduced by a change of variables to an equation of the same type with an unbounded operator whose domain of definition is already constant. Only in ([3,4]) was another type of generalized solutions of problem (1)–(2) also studied in the case when $A(t)$ is a positive-definite self-adjoint operator for which only $D[A^{1/2}(t)]$ is independent of t, and the operator $A^{1/2}(t)A^{-1/2}(0)$ is strongly continuous.

The present note is a continuation of the investigations carried out by M. A. Krasnosel' skii, S. G. Krein, and the author ([6−12]), and is devoted to the study of problem (1)–(2) for the case when the domain of definition of some fractional power of the operator $A(t)$ does not depend on t. An important example of such operators $A(t)$ is given in ([4]).

Let G be a bounded domain with boundary Γ, and

$$A(t)v \equiv -\sum_{i,k=1}^{n} [a_{ik}(t,x)v_{x_k}]'_{x_i} + a(t,x)v \tag{3}$$

be an operator acting in $L_2(G)$, defined on functions $v \in W_2^2(G)$ satisfying the boundary condition

$$v'_{N_t} + \sigma(t,x)v\big|_{\Gamma} = 0, \tag{4}$$

where N_t is the conormal vector. Suppose

$$\sum_{i,k=1}^{n} a_{ik}(t,x)\gamma_i\gamma_k \geq \alpha^2 \sum_{i=1}^{n} \gamma_i^2, \qquad a(t,x) \geq \alpha^2,$$

$$\alpha > 0, \qquad \sigma(t,x) \geq 0.$$

Then $A(t)$ is positive definite and self-adjoint, and

$$D[A^{1/2}(t)] = D[A^{1/2}(0)] = W_2^1(G).$$

O. M. Kozlov ([14]) found sufficient smoothness conditions for the operator $A^{\rho}(t)A^{-\rho}(0)$ for $0 < \rho < 1/2$, and in connection with this the question arose as to what smoothness of this operator ensures the existence of a classical solution of problem (1)–(2).

2. **Theorem 1.** Let $A(t)$ $(0 \leq t \leq 1)$ be a positive-definite self-adjoint operator. Suppose that $D[A^{\rho}(t)]$, for some $\rho \in (0,1)$, does not depend on t, and that the operator $A^{\rho}(t)A^{-\rho}(0)$ satisfies the Lipschitz condition $\mathrm{Lip}(1-\rho+\varepsilon)$ $(0 < \varepsilon \leq \rho)$:

$$\|A^{\rho}(t)A^{-\rho}(0) - A^{\rho}(\tau)A^{-\rho}(0)\| \leq C_1|t-\tau|^{1-\rho+\varepsilon} \quad (0 \leq \tau, t \leq 1). \tag{5}$$

Then there exists an operator $U(t,\tau)$, defined and strongly continuous jointly in t and τ for $0 \leq \tau \leq t \leq 1$. For $t > \tau$ this operator is continuous, in the operator norm, jointly in t and τ, is once continuously differentiable both in t and in τ, and satisfies the equations

$$U'_t(t,\tau) + A(t)U(t,\tau) = 0, \qquad U'_{\tau}(t,\tau) - \overline{U(t,\tau)A(\tau)} = 0 \tag{6}$$

and the initial condition

$$U(t,t) = I. \tag{7}$$

For any $0 \le \tau < t \le 1$, $0 \le \alpha \le \rho$, $0 \le \gamma < 1+\varepsilon$, $\alpha \le \gamma$, the inequalities

$$\|A^{\gamma}(t)U(t,\tau)A^{-\alpha}(\tau)\| \le \frac{C_2(\alpha,\gamma)}{|t-\tau|^{\gamma-\alpha}}, \qquad \left\|\overline{A^{-\alpha}(\tau)U(t,\tau)A^{\gamma}(t)}\right\| \le \frac{C_2(\alpha,\gamma)}{|t-\tau|^{\gamma-\alpha}} \tag{8}$$

hold.

We indicate the scheme of the proof. The operator $U(t,\tau)$ $(0 \le \tau \le t \le 1)$ is constructed as the limit, as $m \to \infty$, of the operators

$$U_m(t,\tau) = e^{-(t_n - t_{n-1})A_n} \dots e^{-(t_1-t_0)A_1}.$$

Here $A_i = A(t_i)$, $t_0 = \tau$, $t_n = t$, and the remaining t_i and n are determined as follows:

$$t_i = \frac{[\tau m] + 1 + i}{m}, \qquad n = [tm] - [\tau m],$$

if the right-hand side of the last equality is ≥ 4; if it is < 4, then $n = 1$. Below, what is mainly used is that the quantities $t_i - t_{i-1}$ are of the same order of smallness.

Let A and B be positive-definite self-adjoint operators and suppose that $D(A^{\rho}) = D(B^{\rho})$. Then

$$e^{-(t-\tau)A} - e^{-(t-\tau)B} = -\sum_{p=1}^{l} \int_{\tau}^{t} A^{(1-\rho)p+\rho_1} e^{-(t-s)A} \Delta_p(A,B) B^{p\rho} e^{-(s-\tau)B}\, ds. \tag{9}$$

Here l is such an integer that $\rho_1 = 1 - l\rho \in (0, \rho]$. If $\rho < \frac{1}{2}$, then $l > 1$,

$$\Delta_p(A,B) = A^{\rho}B^{-\rho} - I$$

for $p = 1, 2, \dots, l-1$, and

$$\Delta_l(A,B) = A^{\rho}B^{-\rho} - \overline{A^{-\rho_1}B^{\rho_1}}.$$

If $\rho \ge \frac{1}{2}$, then $l = 1$ and

$$\Delta_1(A,B) = A^{\rho}B^{-\rho} - \overline{A^{-\rho_1}B^{\rho_1}}.$$

Put

$$U = e^{-(t_k - t_{k-1})A_k} \dots e^{-(t_1-t_0)A_1}.$$

With the aid of (9), the following identities are established:

$$A_k^{q\rho} U A_0^{-\alpha} = A_k^{q\rho} e^{-(t_k - t_1)A_k} e^{-(t_1 - t_0)A_1} A_0^{-\alpha}$$
$$+ \sum_{r=2}^{k-1} \sum_{p=1}^{l} A_k^{1-\rho+q\rho} e^{-(t_k - t_r)A_k} \int_{t_{r-1}}^{t_r} e^{-(t_r - s)A_k} \Delta_p(A_k, A_{r-1}) e^{-(s - t_{r-1})A_{r-1}} \, ds \cdot A_{r-1}^{p\rho} U_{r-1} A_0^{-\alpha}$$
$$- \sum_{r=2}^{k-1} \sum_{p=1}^{l} A_k^{q\rho} e^{-(t_k - t_r)A_k} \int_{t_{r-1}}^{t_r} A_r^{1-p\rho} e^{-(t_r - s)A_r} \Delta_p(A_r, A_{r-1}) e^{-(s - t_{r-1})A_{r-1}} \, ds \cdot A_r^{p\rho} U_{r-1} A_1^{-\alpha}. \quad (10)$$

($k = 3, 4, \ldots, n$), which are a finite-dimensional analogue of one integral identity from (12).

The identities (10) make it possible to estimate (uniformly in m) the norms of the operators $A_k^{q\rho} U_{kA} 0^{-\alpha}$ ($q = 1, \ldots, l$; $0 \le \alpha \le \rho$) and, consequently, $A^\gamma(t) U_m(t, \tau) A^{-\alpha}(\tau)$ ($0 \le \alpha \le \rho$, $0 \le \gamma < 1 + \varepsilon$, $\alpha \le \gamma$), $A^\gamma(t)[U_m(t, \tau) - e^{-(t-\tau)A(t)}] A^{-\alpha}(\tau)$.

Here only the inequalities

$$\|A^\rho(t) A^{-\rho}(\tau)\| \le C_2, \qquad \|\Delta_p[A(t), A(\tau)]\| \le C_3 |t - \tau|^{1-\rho+\varepsilon} \quad (p = 1, \ldots, l),$$

which follow from (5) and the results of (15), are used. The first of these inequalities is a consequence of the second, since always

$$\|A^\rho(t) A^{-\rho}(\tau)\| \le C_4 \|\Delta_p[A(t), A(\tau)]\| + C_5 \quad (p = 1, \ldots, l).$$

The estimates, uniform in m, make it possible to establish that the sequence of operators $A^\gamma(t) U_m(t, \tau)$ ($0 \le \gamma < 1 + \varepsilon$) converges to a limit, which obviously has the form $A^\gamma(t) U(t, \tau)$.

Passing to the limit establishes the first of inequalities (8) and the inequality

$$\|A^\gamma(t)[U(t, \tau) - e^{-(t-\tau)A(t)}] A^{-\alpha}(\tau)\| \le C_6(\gamma, \alpha) |t - \tau|^{1-\rho+\varepsilon-\gamma+\alpha}, \quad (11)$$

which, in particular, shows that the norm of the operator $[U(t, \tau) - e^{-(t-\tau)A(t)}] A^{-\rho}(\tau)$ is an infinitesimal of order > 1 with respect to $t - \tau$. The last fact and the relation $U(t, \tau) = U(t, s) U(s, \tau)$ ($0 \le \tau \le s \le t \le 1$), which follows from the limiting relation $\lim_{m \to \infty} \|\dot{U}_m(t, \tau) - U_m(t, s) U_m(s, \tau)\| = 0$, allow one to derive the first of equations (6).

The second of these equations, as well as the second of inequalities (8), are established analogously. For this it is necessary to consider the operators

$$\widetilde{U}_m(t, \tau) = e^{-(t_n - t_{n-1})A_{n-1}} \ldots e^{-(t_1 - t_0)A_0}$$

and to take into account that

$$\lim_{m\to\infty} \|U_m(t,\tau) - \widetilde{U}_m(t,\tau)\| = 0.$$

With the aid of Theorem 1 one generalizes to the case of an operator with variable domain of definition the existence theorems, cited in (12), for classical and generalized solutions of linear and nonlinear equations.

3. O. M. Kozlov observed that, for concrete differential operators, in the case $\rho = \frac{1}{2}$, instead of estimate (5) it is convenient to verify the same estimate for the operator $A^{1/2}(t)A^{-1/2}(\tau) - A^{-1/2}(t)A^{1/2}(\tau)$. In this case the proof of Theorem 1 is preserved. He established (16), with the aid of extension theory, that the norm of this operator does not exceed $C_7 \max_x |\sigma(t,x) - \sigma(\tau,x)|$, if $A(t)$ is the elliptic operator (3)–(4), whose coefficients do not depend on t. This result can be strengthened.

Theorem 2. Let $A(t)$ be an arbitrary elliptic operator (3)–(4). Then for any $p > n-1$

$$\|A^{1/2}(t)A^{-1/2}(\tau) - A^{-1/2}(t)A^{1/2}(\tau)\| \le C_8(p)\Bigg\{ \max_{x,i,k} |a_{ik}(t,x) - a_{ik}(\tau,x)| + \max_x |a(t,x) - a(\tau,x)| + \left[\int_\Gamma |\sigma(t,x) - \sigma(\tau,x)|^p \, dl\right]^{1/p} \Bigg\}. \tag{12}$$

Proof. Let $z \in D[A(t)]$, and let v be any element of $L_2(G)$.

Obviously,

$$(A(t)z,\ A^{-1/2}(t)v) = \int_G \sum_{i,k=1}^n a_{ik}(t,x)\,\frac{\partial z}{\partial x_k} \cdot \frac{\partial}{\partial x_i} A^{-1/2}(t)v \, d\omega + $$

$$+ \int_G a(t,x) z \cdot A^{-1/2}(t)v \, d\omega + \int_\Gamma \sigma(t,x) z \cdot A^{-1/2}(t) v \, dl.$$

By passage to the limit it is established that, for every $z \in W_2^1(G)$, the right-hand side of this equality is equal to $(A^{1/2}(t)z, v)$. Setting $z = A^{-1/2}(\tau)u$, where u is an arbitrary element of $L_2(G)$, we obtain

$$(A^{1/2}(t)A^{-1/2}(\tau)u,\ v) = \int_G \sum_{i,k=1}^n a_{ik}(t,x)\,\frac{\partial}{\partial x_k} A^{-1/2}(\tau)u \cdot \frac{\partial}{\partial x_i} A^{-1/2}(t) v\, d\omega +$$

$$+\int_G a(t,x)A^{-1/2}(\tau)u\cdot A^{-1/2}(t)v\,d\omega+\int_\Gamma \sigma(t,x)A^{-1/2}(\tau)u\cdot A^{-1/2}(t)v\,dl.$$

Similarly, for any u and v it is shown that

$$(\overline{A^{-1/2}(t)A^{1/2}(\tau)}\,u,\;v)=\int_G\sum_{i,k=1}^n a_{ik}(\tau,x)\,\frac{\partial}{\partial x_k}A^{-1/2}(\tau)u\cdot\frac{\partial}{\partial x_i}A^{-1/2}(t)v\,d\omega+$$

$$+\int_G a(\tau,x)A^{-1/2}(\tau)u\cdot A^{-1/2}(t)v\,d\omega+\int_\Gamma \sigma(\tau,x)A^{-1/2}(\tau)u\cdot A^{-1/2}(t)v\,dl.$$

Comparing the last two equalities, taking into account that

$$\|A^{-1/2}(t)\|\le\frac{1}{\alpha},\qquad \|\operatorname{grad}A^{-1/2}(t)\|\le\frac{1}{\alpha},$$

and applying the embedding theorem of S. L. Sobolev, we obtain (12). Theorem 2 is proved.

With the help of Theorems 1 and 2, existence theorems for solutions of linear and nonlinear parabolic equations with elliptic operator (3)–(4) are established.

Voronezh
Agricultural Institute

Received
24 VII 1958

REFERENCES

1. T. Kato, *J. Math. Soc. Japan*, **5**, No. 2 (1953).
2. M. I. Vishik, *Matem. sborn.*, **39** (81), No. 3 (1956).
3. O. A. Ladyzhenskaya, *Matem. sborn.*, **39** (81), No. 4 (1956).
4. O. A. Ladyzhenskaya, *Matem. sborn.*, **45** (87), No. 2 (1958).
5. M. A. Krasnosel' skii, S. G. Krein, *Tr. 3rd All-Union Math. Congress*, **3**, 1958.

6. M. A. Krasnosel' skii, S. G. Krein, P. E. Sobolevskii, *DAN*, **111**, No. 1 (1956).

7. M. A. Krasnosel' skii, S. G. Krein, P. E. Sobolevskii, *DAN*, **112**, No. 6 (1957).

8. S. G. Krein, P. E. Sobolevskii, *DAN*, **118**, No. 2 (1958).

9. S. G. Krein, *DAN*, **114**, No. 6 (1957).

10. P. E. Sobolevskii, *DAN*, **115**, No. 2 (1957).

11. P. E. Sobolevskii, *DAN*, **116**, No. 5 (1957).

12. P. E. Sobolevskii, *DAN*, **122**, No. 6 (1958).

13. T. Kato, *Div. Electromagn. Res., Inst. Math. Sci. N. Y. Univ., Res. Rep.* No. BR-11 (1955).

14. O. M. Kozlov, *Tr. seminar on functional analysis, Voronezh State Univ.*, No. 6 (1958).

15. E. Heinz, *Math. Ann.*, **123**, 415 (1951).

16. O. M. Kozlov, *DAN*, **123**, No. 6 (1958).

Note: Figure translations are in progress. See original paper for figures.