

# ON THE QUESTION OF EXTREMAL FUNCTIONS IN CERTAIN PROBLEMS OF APPROXIMATION THEORY

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**Abstract**

**Full Text**

**MATHEMATICS**

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**ON THE QUESTION OF EXTREMAL FUNCTIONS IN CERTAIN PROBLEMS OF APPROXIMATION THEORY**

*(Presented by Academician S. L. Sobolev on 25 IV 1958)*

Let  $W^{(r)}H^{(\alpha)}$  be the class of  $2\pi$ -periodic functions having a derivative of order  $r$  satisfying a Lipschitz condition of degree  $\alpha$  with constant 1, and let  $\lambda_k^{(n)}$  ( $k = 0, 1, \dots, n + 1$ ;  $\lambda_0^{(n)} = 1$ ;  $\lambda_{n+1}^{(n)} = 0$ ) be a triangular matrix of numbers satisfying the conditions

$$\mu_k^{(n)} \leq \mu_{k+1}^{(n)}, \quad \Delta^2 \mu_k^{(n)} \geq 0, \tag{1}$$

where

$$\mu_k^{(n)} = \frac{1 - \lambda_k^{(n)}}{k^2}.$$

For each function  $f(x) \in W^{(r)}H^{(\alpha)}$  we form the sequence of trigonometric polynomials of order  $n$

$$U_n(f; x; \lambda) = \frac{a_0}{2} + \sum_{k=1}^n \lambda_k^{(n)} (a_k \cos kx + b_k \sin kx),$$

where  $a_k$  and  $b_k$  are the Fourier coefficients of the function  $f(x)$ , and we investigate the behavior of the absolute value of the deviation  $f(x) - U_n(f; x; \lambda)$ .

A. F. Timan showed <sup>(3)</sup> that for any function  $f(x) \in W^{(r)}H^{(\alpha)}$ , where  $r \geq 0$  is any integer and  $0 \leq \alpha < 1$ , as  $n \rightarrow \infty$  the asymptotic inequality

$$\begin{aligned} & |f(x) - U_n(f; x; \lambda)| \leq \\ & \leq \frac{2^{\alpha+1}}{\pi^2 n^\alpha} \int_0^{\pi/2} t^\alpha \sin t \, dt \cdot \left| \sum_{k=1}^n \frac{\mu_k^{(n)}}{n - k + 1} - \frac{\ln n}{n^r} \right| + O\left(\frac{1}{n^{r+\alpha}}\right) \end{aligned} \tag{2}$$

holds (in the case  $\alpha = 0$  the right-hand side must be doubled), and moreover for every  $n$  there exists a function  $\varphi_n(x)$  for which inequality (2) becomes an asymptotic equality.

However, the question remains open: does there exist in the class  $W^{(r)}H^{(\alpha)}$  a function  $f(x)$ , independent of  $n$ , for which inequality (2) would become an asymptotic equality at least along some sequence of values of  $n$ . Such a question was first considered by S. M. Nikol'skii<sup>(2)</sup> (as applied to polynomials of best approximation). The purpose of the present work is to solve this question for linear methods of approximation.

**Theorem 1.** *If the matrix of numbers  $\lambda_k^{(n)}$  satisfies conditions (1), then for each value  $x_0 \in [-\pi; \pi]$  there exists a function  $f_0(x) \in W^{(r)}H^{(\alpha)}$  ( $0 \leq \alpha < 1$ ,  $r \geq 0$  an integer), for which the inequalities hold*

$$\begin{aligned} & |f_0(x_0) - U_{n_i}(f_0; x_0; \lambda)| > \\ & > \frac{2^{\alpha+1}}{\pi^2 n_i^\alpha} \int_0^{\pi/2} t^\alpha \sin t \, dt \cdot \left| \sum_{k=1}^{n_i} \frac{\mu_k^{(n_i)}}{n_i - k + 1} - \frac{\ln n_i}{n_i^r} \right| (1 - \varepsilon_{n_i}), \end{aligned} \quad (3)$$

where  $n_i$  ( $i = 1, 2, \dots$ ) is a certain increasing sequence of natural numbers and  $\varepsilon_{n_i} \rightarrow 0$  as  $i \rightarrow \infty$ .

For the cases  $\lambda_k^{(n)} = 1$  ( $k = 0, 1, \dots, n$ ),  $r = 0$ ,  $0 < \alpha \leq 1$ , and  $\lambda_k^{(n)} = 1$  ( $k = 0, 1, \dots, n$ ),  $r = 1, 2, \dots$ ,  $\alpha = 0$ , this result was obtained by G. Ya. Doronina<sup>(1)</sup>.

The proof of Theorem 1 is based on the following lemma.

**Lemma.** *If the matrix of numbers  $\lambda_k^{(n)}$  satisfies conditions (1), then for every  $0 \leq \alpha < 1$  and integer  $r \geq 0$  the relations*

$$\begin{aligned} \frac{1}{\pi} \int_a^\pi f(t) L_{nr}(t) \, dt &= O\left(\frac{1}{n^{r+\alpha}}\right), \\ \frac{1}{\pi} \int_0^{C/n} f(t) L_{nr}(t) \, dt &= O\left(\frac{1}{n^{r+\alpha}}\right), \end{aligned}$$

hold, where  $0 < a \leq \pi$  and  $C$  are fixed numbers;

$$L_{nr}(t) = \sum_{k=0}^n \mu_k^{(n)} \cos\left(kt + \frac{r\pi}{2}\right) + \sum_{k=n+1}^{\infty} \frac{\cos(kt + r\pi/2)}{k^r}.$$

The equalities (4) hold uniformly over all functions  $f(x) \in W^{(r)}H^{(\alpha)}$ .

For the proof of the theorem, consider the function  $\varphi_n(x)$ , defined on  $[0, \pi]$  by the equalities

$$\varphi_n(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq x_1^{(n)}; \\ & x_{n-1}^{(n)} \leq x \leq \pi \text{ for even } n, \\ & x_{n-2}^{(n)} \leq x \leq \pi \text{ for odd } n; \\ (-1)^\nu 2^{\alpha-1} (x - x_\nu^{(n)})^\alpha, & \text{if } x_\nu^{(\alpha)} \leq x \leq x_{\nu+1/2}^{(n)}, \\ & \nu = 1, 2, \dots, n-2 \text{ for even } n, \\ & \nu = 1, 2, \dots, n-3 \text{ for odd } n; \\ (-1)^\nu 2^{\alpha-1} (x_{\nu+1}^{(\alpha)} - x)^\alpha, & \text{if } x_{\nu+1/2}^{(n)} \leq x \leq x_{\nu+1}^{(n)}, \end{cases}$$

where

$$x_\nu^{(n)} = \frac{\nu + l - r/2}{n} \pi,$$

and  $l$  is the smallest integer greater than  $r/2$ .

On the basis of the lemma just stated, one establishes the possibility of choosing an increasing sequence of natural numbers  $\{n_i\}$  such that, uniformly over all functions  $\varphi(x) \in W^{(r)}H^{(\alpha)}$ , the inequalities

$$\begin{aligned} & \left| \frac{1}{\pi} \int_0^{x_1^{(n_i-1)}} \varphi_{n_i}(t) L_{n_i r}(t) dt \right| > \\ & > \frac{2^\alpha}{\pi^2 n_i^\alpha} \int_0^{\pi/2} t^\alpha \sin t dt \cdot \left| \sum_{k=1}^{n_i} \frac{\mu_k^{(n_i)}}{n_i - k + 1} - \frac{\ln n_i}{n_i^r} \right| \left( 1 - \frac{\varepsilon_{n_i}}{3} \right); \end{aligned}$$

$$\frac{1}{\pi} \left| \int_{x_1^{(n_i-1)}}^\pi \varphi(t) L_{n_i r}(t) dt \right| < \frac{2^\alpha}{\pi^2 n_i^\alpha} \int_0^{\pi/2} t^\alpha \sin t dt \cdot \left| \sum_{k=1}^{n_i} \frac{\mu_k^{(n_i)}}{n_i - k + 1} - \frac{\ln n_i}{n_i^r} \right| \frac{\varepsilon_{n_i}}{3},$$

$$\frac{1}{\pi} \left| \int_0^{x_1^{(n_i)}} \varphi(t) L_{n_i r}(t) dt \right| < \frac{2^\alpha}{\pi^2 n_i^\alpha} \int_0^{\pi/2} t^\alpha \sin t dt \left| \sum_{k=1}^{n_i} \frac{\mu_k^{(n_i)}}{n_i - k + 1} - \frac{\ln n_i}{n_i^r} \right| \frac{\varepsilon_{n_i}}{3},$$

where  $\{\varepsilon_n\}$  is a certain null-sequence satisfying the condition

$$\overline{\lim}_{n \rightarrow \infty} \varepsilon_n \left| \sum_{k=1}^n \frac{\mu_k^{(n)}}{n - k + 1} - \frac{\ln n}{n^r} \right| = \infty$$

and the  $2\pi$ -periodic function  $\varphi_r^*(x)$  is considered, defined on  $[-\pi, \pi]$  by the equalities

$$\varphi_r^*(x) = \begin{cases} 0, & \text{if } x_1^{(n_i)} \leq x \leq x_{\nu_0}^{(n_i)}; \\ \varphi_{n_i}(x), & \text{if } x_{\nu_0}^{(n_i)} \leq x \leq \beta_{i-1} \quad (i = 1, 2, \dots); \\ 0, & \text{if } \beta_{i-1} \leq x \leq x_1^{(n_{i-1})}; \end{cases}$$

$$\varphi_r^*(0) = 0;$$

$$\varphi_r^*(-x) = \varphi_r^*(x), \quad \text{if } r \text{ is even};$$

$$\varphi_r^*(-x) = -\varphi_r^*(x), \quad \text{if } r \text{ is odd}.$$

Here  $\beta_{i-1}$  is the zero of the function  $\varphi_{n_i}(x)$  nearest on the left to  $x_1^{(n_{i-1})}$  such that

$\varphi_{n_i}(\beta_{i-1} - \pi/2n_i)$ ;  $\nu$  is the smallest odd number satisfying the condition  $\nu_0 > 2^{1/\alpha-1} + 1$ .

The function  $f_0(x)$ , defined for the given  $x_0$  by the relation

$$f_0^{(r)}(x_0 + x) = \varphi_r^*(x),$$

belongs to the class  $W^{(r)}H^{(\alpha)}$  and satisfies the inequalities (3).

From Theorem 1 it follows:

**Theorem 2.** In order that the linear method of approximation  $U_n(f; x; \lambda)$ , as  $n \rightarrow \infty$ , should realize, for every function  $f(x) \in W^{(r)}H^{(\alpha)}$ , approximation of the same order as the best approximation, it is necessary that

$$\sum_{k=1}^n \frac{\mu_k^{(n)}}{n-k+1} = \frac{\ln n}{n^r} + O\left(\frac{1}{n^r}\right).$$

Theorem 2 supplements Theorem 4 of work <sup>1</sup>.

I consider it my duty to express my deep gratitude to A. F. Timan for his help in carrying out the present work.

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## REFERENCES

- <sup>1</sup> G. Ya. Doronin, DAN, 69, No. 4 (1949).
- <sup>2</sup> S. M. Nikol'skii, Izv. AN SSSR, ser. matem., 10, 393 (1946).
- <sup>3</sup> A. F. Timan, Izv. AN SSSR, ser. matem., 17, No. 2 (1953).

*Note: Figure translations are in progress. See original paper for figures.*

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