MATHEMATICS

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Abstract

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ON A PRINCIPLE OF EXISTENCE OF BOUNDED, PERIODIC, AND ALMOST-PERIODIC SOLUTIONS OF A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

(Presented by Academician P. S. Aleksandrov on 9 VI 1958)

1. Consider the system of differential equations

$$\frac{dx_i}{dt} = f_i(t, x_1, \dots, x_n) \quad (i = 1, \dots, n), \tag{1}$$

where we shall assume that the right-hand sides are continuous in the aggregate of variables $-\infty < t,\ x_1, \dots, x_n < +\infty$. We shall write system (1) in vector form

$$\frac{dx}{dt} = f(t, x). \tag{2}$$

Let two continuously differentiable functions $\lambda(x)$ and $\mu(x)$ be given on the n-dimensional space E^n. Suppose that

$$(f(t, x), \operatorname{grad} \lambda(x)) = \sum_{i=1}^{n} f_i \frac{\partial \lambda}{\partial x_i} > 0 \tag{3}$$

for $\|x\| > R$, where R is some positive number. Let

$$m = \min_{\|x\|=R} \lambda(x), \qquad M = \max_{\|x\|=R} \lambda(x). \tag{4}$$

Suppose that on the set T of those $x \in E^n$ for which $m \leq \lambda(x) \leq M$, $\|x\| > R$, the function $\mu(x)$ satisfies the condition

$$(f(t, x), \operatorname{grad} \lambda(x) + \operatorname{grad} \mu(x)) = \sum_{i=1}^{n} f_i \left(\frac{\partial \lambda}{\partial x_i} + \frac{\partial \mu}{\partial x_i} \right) > 0, \tag{5}$$

and

$$\lim_{x\in T,\ \|x\|\to+\infty} |\mu(x)| = +\infty. \tag{6}$$

If the listed conditions are fulfilled, we shall say that system (1) satisfies condition $(*)$ with respect to the functions $\lambda(x)$ and $\mu(x)$.

Theorem 1. *Let system (1) satisfy condition $(*)$ with respect to the functions $\lambda(x)$ and $\mu(x)$, and let $\lambda(x)$ be even: $\lambda(-x) = \lambda(x)$.*

Then system (1) has at least one solution defined on $(-\infty, +\infty)$ and uniformly bounded.

If the right-hand sides of system (1) are periodic in t, then system (1) has at least one periodic solution with the same period.

If the right-hand sides of system (1) are almost-periodic in t (uniformly in every ball), then system (1) has at least one almost-periodic solution.

Let us emphasize that the hypotheses of this theorem do not assume the uniqueness conditions for solutions of system (1) to be satisfied, and do not assume that all solutions of system (1) are nonlocally continuable.

2. As an example, consider a system of the form

$$\frac{dx}{dt} = Ax + \varphi(t, x), \tag{7}$$

where A is a matrix all of whose eigenvalues have real parts different from zero. Without loss of generality (otherwise one can make a change of variables) one may assume that system (7) has the form

$$\frac{dy}{dt} = A_1 y + \varphi_1(t, y, z),$$

$$\frac{dz}{dt} = -A_2 z + \varphi_2(t, y, z), \tag{8}$$

where y is a k-dimensional vector ($k \le n$), z is an $(n-k)$-dimensional vector, and the matrices A_1 and A_2 have eigenvalues with positive real parts. Let B_1 and B_2 be symmetric positive-definite matrices of orders k and $(n-k)$, respectively, such that

$$(B_1 y, A_1 y) \ge \|y\|^2, \qquad (B_2 z, A_2 z) \ge \|z\|^2.$$

The existence of the matrices B_1 and B_2 is proved, for example, in Lyapunov's theory of stability of motion (^{9}).

If we put $\lambda(x) = \lambda(y + z) = \frac{1}{2}(B_1 y, y) - \frac{1}{2}(B_2 z, z)$ and $\mu(x) = \varepsilon\|x\|^2$, where $\varepsilon > 0$ is sufficiently small, then system (8) satisfies condition (*) with respect to these functions if, for $\|x\| \geq r$, the inequalities

$$(\varphi_1(t, y, z), B_1 y) - (\varphi_2(t, y, z), B_2 z) > q(\|y\|^2 + \|z\|^2), \tag{9}$$

$$(\varphi_1(t, y, z), y) + (\varphi_2(t, y, z), z) > -p(\|y\|^2 + \|z\|^2), \tag{10}$$

hold, where $0 < q < 1$ and p is an arbitrary fixed number.

Thus, the fulfillment of inequalities (9)–(10) for large $\|x\|$ guarantees the existence of a bounded solution. They guarantee the existence of a periodic solution if the right-hand sides are periodic in t, and the existence of an almost-periodic solution if the right-hand sides are almost-periodic. Conditions (9)–(10) are satisfied, in particular, if $\|\varphi(t, x)\| = o(\|x\|)$ as $\|x\| \to +\infty$. In this case we arrive at a theorem proved by B. P. Demidovich (^{6}).

As a second example, consider the second-order system

$$\frac{dx}{dt} = Ax^3 + 3Bx^2 y + Cxy^2 + Dy^3 + \varphi_1(t, x, y),$$

$$\frac{dy}{dt} = Bx^3 + Cx^2 y + 3Dxy^2 + Ey^3 + \varphi_2(t, x, y), \tag{11}$$

where

$$|\varphi_1(t, x, y)| = o(|x|^3 + |y|^3), \qquad |\varphi_2(t, x, y)| = o(|x|^3 + |y|^3),$$

and moreover $|A| + |B| > 0$, and the cubic equations

$$Ak^3 + 3Bk^2 + Ck + D = 0, \qquad Bk^3 + Ck^2 + 3Dk + E = 0$$

have no common real roots.

It can be shown that, under these assumptions, system (11) satisfies the conditions of Theorem 1 if, as $\lambda(x, y)$ and $\mu(x, y)$, one takes respectively the functions

$$\lambda(x, y) = Ax^4 + 4Bx^3 y + 2Cx^2 y^2 + 4Dxy^3 + Ey^4, \qquad \mu(x, y) = x^2 + y^2.$$

3. We shall say that the function $\lambda(x)$ is nondegenerate if its gradient does not vanish outside some ball and if the vector field $\operatorname{grad} \lambda(x)$ has nonzero rotation (^{1,7}) on spheres of sufficiently large radius centered at the origin.

The function $\lambda(x)$ appearing in the hypotheses of Theorem 1 is nondegenerate, since $\|\operatorname{grad}\lambda(x)\| > 0$ for $\|x\| \geq R$, and since $\operatorname{grad}\lambda(x)$ is odd for an even function $\lambda(x)$; hence the rotation is odd (by the theorem of Shnirelman–Lyusternik–Borsuk ([3,7,8])).

The concept of a nondegenerate function makes it possible to formulate an assertion stronger than Theorem 1.

Theorem 2. *Let system (1) satisfy condition* () with respect to the functions $\lambda(x)$ and $\mu(x)$, and let the function $\lambda(x)$ be nondegenerate.*

Then system (1) has at least one solution, defined on $(-\infty, +\infty)$, *that is uniformly bounded.*

If the right-hand sides of system (1) are periodic in t, then system (1) has at least one periodic solution with the same period.

If the right-hand sides of system (1) are almost-periodic in t, then system (1) has at least one almost-periodic solution.

To apply Theorem 2 one must be able to compute the rotation of the vector field $\operatorname{grad}\lambda(x)$. In the general case this problem is difficult. In the case when the function $\lambda(x)$ is periodic ($\lambda(Ux) \equiv \lambda(x)$, where $U^p = I$), to compute the rotation one may use the theorems from (8). In the case of a two-dimensional space the problem is simplified. For example, in the (x, y)-plane the function $\lambda(x, y) = (\alpha_1 x + \beta_1 y) \cdots (\alpha_{nx} + \beta_{ny})$ $(|\alpha_i| + |\beta_i| > 0)$ is nondegenerate, and the rotation of the gradient field is equal to $-n + 1$, if all the straight lines $\alpha_{ix} + \beta_{iy} = 0$ are distinct.

Consider the system

$$\begin{aligned} \frac{dx}{dt} &= Ax^2 + 2Bxy + Cy^2 + \varphi_1(t, x, y), \\ \frac{dy}{dt} &= Bx^2 + 2Cxy + Dy^2 + \varphi_2(t, x, y), \end{aligned} \tag{12}$$

where $|\varphi_1| + |\varphi_2| = o(x^2 + y^2)$ as $x^2 + y^2 \to +\infty$, and

$$\begin{vmatrix} A & B \\ B & C \end{vmatrix} \begin{vmatrix} B & C \\ C & D \end{vmatrix} - \frac{1}{4} \begin{vmatrix} A & C \\ B & D \end{vmatrix}^2 > 0. \tag{13}$$

Under these assumptions system (12) satisfies the conditions of Theorem 2, if one sets

$$\lambda(x, y) = Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3, \qquad \mu(x, y) = x^2 + y^2.$$

4. In some cases conditions (3), (5) are fulfilled only in a certain spherical shell $R \leq \|x\| \leq R_1$. The assertions of Theorems 1 and 2 then remain valid if, instead of condition (6), the inequality

$$|\mu(x) - \mu(x^*)| \geq M - m$$

is fulfilled for $\|x\| = R$, $\|x^*\| = R_1$ and $m \leq \lambda(x)$, $\lambda(x^*) \leq M$.

As a consequence of this fact one can obtain certain theorems on the existence of periodic and almost-periodic solutions for equations with a small parameter (for example, the theorems established by G. I. Biryuk[2]).

A second consequence is a theorem on the existence, for sufficiently small ε, of a solution defined on $(-\infty, +\infty)$ and bounded for the system

$$\frac{dx}{dt} = Ax + f_1(t,x) + f_2(t,x,\varepsilon),$$

where the constant matrix A has eigenvalues with nonzero real parts, if the conditions

$$\lim_{r \to 0} \sup_{\substack{-\infty<t<+\infty \\ \|x\|<r}} \frac{\|f_1(t,x)\|}{\|x\|} = 0, \qquad \lim_{\varepsilon \to 0} \sup_{\substack{-\infty<t<+\infty \\ \|x\|<r_0}} \|f_2(t,x,\varepsilon)\| = 0$$

are satisfied.

5. The proofs of Theorems 1 and 2 use a simple topological consideration which, in a special case, was applied by Halanaĭ and Beršteĭn (10). We shall formulate this assertion in the form of a theorem.

Theorem 3. Suppose that on the boundary Γ of a bounded domain $G \subset E^n$ the vector fields $f(t,x)$ $(-\infty < t < +\infty)$ have nonzero rotation.

Suppose that for system (2) the nonlocal existence theorems and uniqueness theorems (see ([5],[9])) for the solution are valid.

Suppose that every solution $x(t)$, $(t \in [T_1, T_2])$, of system (2), satisfying the initial condition $x(T_1) \in \Gamma$, satisfies the condition of nonreturn

$$x(t) \neq x(T_1) \quad (t \in (T_1, T_2]). \tag{14}$$

Then system (2) has at least one solution $x^*(t)$ such that

$$x^*(T_2) = x^*(T_1) \in G. \tag{15}$$

Let us emphasize that in condition (1) it is not assumed that $T_1 \ll T_2$. We note that in Theorem 3 it is sufficient to assume that the rotation of the field $f(T_1, x)$ is nonzero.

Theorem 3 is directly applicable to the proof of the existence of periodic solutions. To prove the existence of solutions bounded for all t, it is natural, in the hypotheses of Theorem 3, to let T_1 tend to $-\infty$, and T_2 to $+\infty$. If, in this process, the solutions satisfying condition (15) are uniformly bounded, then some subsequence of them converges to a solution bounded for all t. In Theorems 1 and 2 the uniform boundedness of the solutions satisfying condition (15) follows from certain estimates for the increments of the functions $\lambda(x)$ and $\mu(x)$. A sufficient condition for the existence of bounded solutions satisfying (15) can be formulated as the assumption that the positive and negative semitrajectories of the points of the boundary Γ do not intersect one another.

We note that many assertions following from Theorem 3 can also be obtained by using the well-known topological principle of Ważewski (4).

6. Let us consider the differential equation (2) in a Banach space (see (9)). The method set forth above has so far been successfully applied only to the proof of theorems on the existence of periodic solutions. Moreover, at present one can consider only the case in which the operator $f(t, x)$ is continuous. As an application of the theorems on the existence of periodic solutions of equations in Banach spaces, one can indicate various sufficient conditions for the existence of periodic solutions in certain classes of nonlinear integro-differential equations.

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Note: Figure translations are in progress. See original paper for figures.

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